

odds-inversion.nb Table of Contents

Beyond Varsity Math:

The red-and-blue-balls puzzle

An odds inversion problem

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