CISC 1400
Discrete Structures
Chapter 1
Sets

Arthur G. Werschulz

Fordham University Department of Computer and Information Sciences

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Outline

Sets

- *Set*: a collection of objects (the *members* or *elements* of the set)
- Set-lister notation: curly braces around a list of the elements
  - $\{a, b, c, d, e, f\}$
  - $\{\text{Arizona}, \text{California}, \text{Massachusetts}, 42, 47\}$
- A set may contain other sets as elements:

$$\Big\{1, 2, \{1, 2\}\Big\}$$

- The empty set $\emptyset = \{\}$ contains no elements
- Can use variables (usually upper case letters) to denote sets

$$C = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$$

- *Universal set* (generally denoted U): contains all elements we might ever consider (only consider what matters)

Enumerating the elements of a set

- Order doesn't matter

$$\{1, 2, 3\} = \{3, 1, 2\}$$

- Repetitions don't count

$$\{a, b, b\} = \{a, b\}$$

  (better yet: don't repeat items in a listing of elements)

Element notation

If A is a set, then

- $x \in A$ means "x is an element of A"
- $x \notin A$ means "x is not an element of A"

So

- $e \in \{a, e, i, o, u\}$
- $f \notin \{a, e, i, o, u\}$

Some well-known sets

- Pretty much standard notations:
  - $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$: the set of *natural numbers* (non-negative integers).
  - $\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, \dots\}$: the set of all *integers*.
  - $\mathbb{Q}$: the set of all *rational numbers* (fractions).
  - $\mathbb{R}$: the set of all *real numbers*.
- Less standard (but useful) notations:
  - $\mathbb{Z}^+$ is the set of positive integers.
  - $\mathbb{Z}^-$ is the set of negative integers.
  - $\mathbb{Z}^{\geq 0}$ is the same as $\mathbb{N}$.
  - $\mathbb{R}^{>7}$ is the set of all real numbers greater than seven.

Set builder notation

Rather than listing all the elements

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

(inconvenient or essentially impossible), describe sets via a *property*

$$A = \{x : p(x) \text{ is true}\}$$

Examples:

$$\mathbb{N} = \{x \mid x \in \mathbb{Z} \text{ and } x \geq 0\}$$
$$\mathbb{N} = \{x : x \in \mathbb{Z} \text{ and } x \geq 0\}$$
$$\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$$
$$\mathbb{N} = \{x \in \mathbb{Z} : x \geq 0\}$$

Set builder notation (cont'd)

More examples:

$$\{x \in \mathbb{Q} : 2x = 7\} = \{3.5\}$$
$$\{x \in \mathbb{Z} : 2x = 7\} = \emptyset$$
$$\{2x \mid x \in \mathbb{Z}\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$$
$$\{x \in \mathbb{N} : \tfrac{1}{3}x \in \mathbb{Z}\} = \{0, 3, 6, 9, \dots\}$$
$$\{x \in \mathbb{R}^{\geq 0} : x^2 = 2\} = \{\sqrt{2}\}$$
$$\{x \in \mathbb{Q}^{\geq 0} : x^2 = 2\} = \emptyset$$

Comparison relations on sets

- A is a *subset* of B (written "$A \subseteq B$") if each element of A is also an element of B.
  - $\{1,3,5\} \subseteq \{1,2,3,4,5\}$
  - $\{1,2,3,4,5\} \subseteq \{1,2,3,4,5\}$
  - $\{1,3,6\} \not\subseteq \{1,2,3,4,5\}$
  - $A \subseteq A$ for *any* set A
  - $\emptyset \subseteq A$ for *any* set A
- A is a *proper subset* of B (written "$A \subset B$") if $A \subseteq B$ *and* $A \neq B$.
  - $\{1,3,5\} \subset \{1,2,3,4,5\}$
  - $\{1,2,3,4,5\} \not\subset \{1,2,3,4,5\}$
- $\subset$ vs. $\subseteq$ is somewhat like $<$ vs. $\leq$

Element vs. subset

- $\in$ means "is an element of"
- $\subseteq$ means "is a subset of"

$$A \subseteq B \qquad \text{means} \qquad \text{if } x \in A \text{ then } x \in B$$

- **Examples:** Let

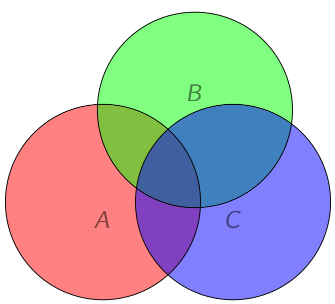
$$A = \{\text{purple}, \text{blue}, \text{orange}, \text{red}\} \qquad \text{and} \qquad B = \{\text{blue}\}.$$

Fill in the missing symbol from the set $\{\in, \notin, \subseteq, \subset, \not\subseteq, =, \neq\}$ to correctly complete each of the following statements:

$$
\begin{array}{rcl}
B \;\_\_\_\; A & & \notin, \subseteq, \subset, \neq \\
\text{blue} \;\_\_\_\; A & & \in, \not\subseteq, \neq \\
\text{green} \;\_\_\_\; A & & \notin, \not\subseteq, \neq \\
A \;\_\_\_\; A & & \notin, \subseteq, = \\
\{\text{blue}, \text{purple}\} \;\_\_\_\; B & & \notin, \not\subseteq, \neq \\
A \;\_\_\_\; B & & \notin, \not\subseteq, \neq
\end{array}
$$

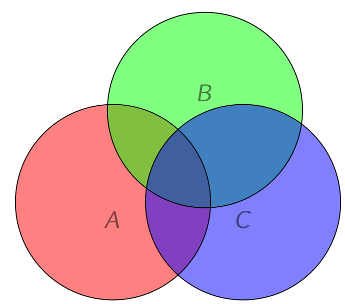
Venn Diagram

Diagram for visualizing sets and set operations



When necessary, indicate universal set via rectangle surrounding the set circles.

Venn Diagram (cont'd)



For example, might have

$$
\begin{aligned}
A &= \{\text{Fordham students who've taken CISC 1400}\} \\
B &= \{\text{Fordham students who've taken CISC 1600}\} \\
C &= \{\text{Fordham students who've taken ECON 1100}\}
\end{aligned}
$$

Set operations: Cardinality

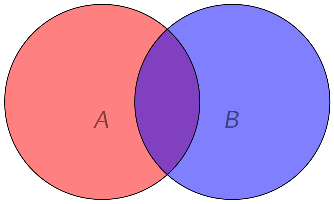
The number of elements in a set is called its *cardinality*.
We denote the cardinality of S by $|S|$.

- Let $A = \{a, b, c, d, e, z\}$. Then $|A| = 6$.
- $|\{a, e, i, o, u\}| = 5$.
- $|\emptyset| = 0$.
- $|\{a, \{b, c\}, d, \{e, f, g\}, h\}| = 5$.

Set operations: Union

Set of all elements belonging to *either* of two given sets:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



Set operations: Union (examples)

Let

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{0, 2, 4, 6, 8\}$$
$$C = \{0, 5, 10, 15\}$$

Then

$$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8\}$$
$$B \cup A = \{0, 1, 2, 3, 4, 5, 6, 8\}$$
$$B \cup C = \{0, 2, 4, 5, 6, 8, 10, 15\}$$
$$(A \cup B) \cup C = \{0, 1, 2, 3, 4, 5, 6, 8, 10, 15\}$$

Let

$$L = \{e, g, b, d, f\}$$
$$S = \{f, a, c, e\}$$

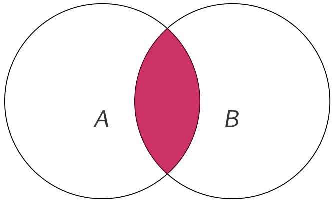
Then

$$L \cup S = \{a, b, c, d, e, f, g\}$$

Set operations: Intersection

Set of all elements belonging to *both* of two given sets:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



Note: We say that two sets are *disjoint* if their intersection is empty.

Set operations: Intersection (examples)

Let

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{0, 2, 4, 6, 8\}$$
$$C = \{0, 5, 10, 15\}$$

Then

$$A \cap B = \{2, 4\}$$
$$B \cap A = \{2, 4\}$$
$$B \cap C = \{0\}$$
$$(A \cap B) \cap C = \emptyset$$

Let

$$L = \{e, g, b, d, f\}$$
$$S = \{f, a, c, e\}$$

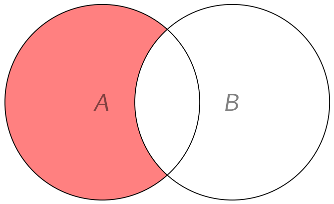
Then

$$L \cap S = \{e, f\}$$

Set operations: Difference

Set of all elements belonging to one set, but not another:

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$



Note that

$$|A - B| = |A| - |A \cap B|$$

Set operations: Difference (examples)

Let

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{0, 2, 4, 6, 8\}$$
$$C = \{0, 5, 10, 15\}$$

Then

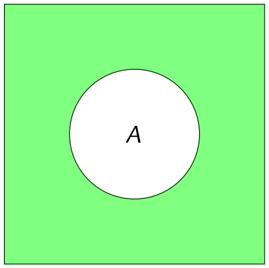
$$A - B = \{1, 3, 5\}$$
$$B - A = \{0, 6, 8\}$$
$$B - C = \{2, 4, 6, 8\}$$
$$C - B = \{5, 10, 15\}$$
$$(A - B) \cap (B - A) = \emptyset \qquad \text{(Are you surprised by this?)}$$

Set operations: Complement

Set of all elements (of the universal set) that do *not* belong to a given set:

$$A' = U - A.$$

Venn diagrams dealing with complements generally use a surrounding rectangle to indicate the universal set U:



If U and A are finite sets, then

$$|A'| = |U - A| = |U| - |A|.$$

Set operations: Complement (examples)

- Let

$$U = \{\text{red}, \text{orange}, \text{yellow}, \text{green}, \text{blue}, \text{indigo}, \text{violet}\}$$
$$P = \{\text{red}, \text{green}, \text{blue}\}$$

  Then

$$P' = \{\text{orange}, \text{yellow}, \text{indigo}, \text{violet}\}$$

- Let E and O respectively denote the sets of even and odd integers. Suppose that our universal set is $\mathbb{Z}$. Then

$$E' = O$$
$$O' = E$$

Set operations: Power Set

Set of all subsets of a given set

$$B \in \mathscr{P}(A) \text{ if and only if } B \subseteq A$$

$$\mathscr{P}(\emptyset) = \{\emptyset\}$$
$$\mathscr{P}(\{a\}) = \big\{\emptyset, \{a\}\big\}$$
$$\mathscr{P}(\{a, b\}) = \big\{\emptyset, \{a\}, \{b\}, \{a, b\}\big\}$$
$$\mathscr{P}(\{a, b, c\}) = \big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\big\}$$

How many elements does $\mathscr{P}(A)$ have?

$$|\mathscr{P}(A)| = 2^{|A|},$$

i.e.,

$$\text{if } |A| = n, \text{ then } |\mathscr{P}(A)| = 2^n.$$

Some basic laws of set theory

Here, U is a universal set, with $A, B, C, S \subseteq U$.

Name	Law
Identity	$S \cap U = S$
Identity	$S \cup \emptyset = S$
Complement	$S \cap S' = \emptyset$
Complement	$S \cup S' = U$
Double Complement	$(S')' = S$
Idempotent	$S \cap S = S$
Idempotent	$S \cup S = S$
Commutative	$A \cap B = B \cap A$
Commutative	$A \cup B = B \cup A$

Some basic laws of set theory (cont'd)

Once again, U is a universal set, with $A, B, C, S \subseteq U$.

Name	Law
Associative	$(A \cap B) \cap C = A \cap (B \cap C)$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
DeMorgan	$(A \cap B)' = A' \cup B'$
DeMorgan	$(A \cup B)' = A' \cap B'$
Equality	$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
Transitive	if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Set operations: Cartesian Product

- *Ordered pair*: Pair of items, in which order matters.
  - $(1,2)\ldots$ not the same thing as $(2,1)$
  - $(\text{red}, \text{blue})$
  - $(1, \text{green})$
- *Cartesian product* (also known as *set product*): Set of all ordered pairs from two given sets, i.e.,

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$$

Set operations: Cartesian Product (examples)

Let

$$\begin{aligned} A &= \{1,2,3\} \\ B &= \{a,b,c\} \\ C &= \{-1,5\}, \end{aligned}$$

Then ...

$$\begin{aligned} A \times B &= \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\} \\ B \times A &= \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\} \\ C \times A &= \{(-1,1),(-1,2),(-1,3),(5,1),(5,2),(5,3)\} \\ B \times C &= \{(a,-1),(a,5),(b,-1),(b,5),(c,-1),(c,5)\} \end{aligned}$$

Note the following:

- $A \times B \neq B \times A$ (unless $A = B$)
- $|A \times B| = |A| \cdot |B|$ (that's why it's called "product").

Principle of Inclusion/Exclusion

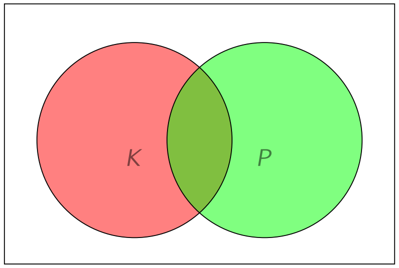
Suppose you run a fast-food restaurant. After doing a survey, you find that

- 25 people like ketchup on their hamburgers,
- 35 people like pickles on their hamburgers,
- 15 people like both ketchup *and* pickles on their hamburgers.

How many people like either ketchup *or* pickles (maybe both) on their hamburgers?
Let $K = \{\text{people who like ketchup}\}$ and $P = \{\text{people who like pickles}\}$. Then

$$|K| = 25 \qquad |P| = 35 \qquad |K \cap P| = 15.$$

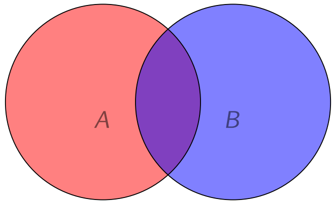
Principle of Inclusion/Exclusion



Since we don't want to count $K \cap P$ twice, we have

$$|K \cup P| = |K| + |P| - |K \cap P| = 25 + 35 - 15 = 45.$$

Set operations: cardinalities of union and intersection



- *Inclusion/exclusion principle:*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

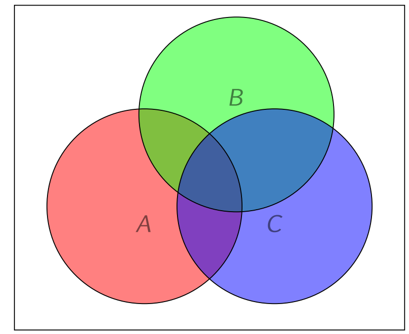
- If A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

- See the example that animates this concept.
- What about three sets (hamburger eaters who like ketchup, pickles, and tomatoes)?

Principle of Inclusion/Exclusion

For three sets:



$$|A \cup B \cup C| = \\ |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Principle of Inclusion/Exclusion

Let K, P, T represent the sets of people who like ketchup, pickles, and tomatoes on their hamburgers. Suppose that

$$|K| = 20 \qquad |P| = 30 \qquad |T| = 45$$
$$|K \cap P| = 10 \qquad |K \cap T| = 12 \qquad |P \cap T| = 13$$
$$|K \cap P \cap T| = 8.$$

Then

$$\begin{aligned} |K \cup P \cup T| &= |K| + |P| + |T| - \\ &\quad |K \cap P| - |K \cap T| - |P \cap T| + \\ &\quad |K \cap P \cap T| \\ &= 20 + 30 + 45 - 10 - 12 - 13 + 8 \\ &= 68 \end{aligned}$$