CISC 1400
Discrete Structures
Chapter 8
Algorithms

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What is an algorithm?

- There are many ways to define an algorithm
  - An algorithm is a step-by-step procedure for carrying out a task or solving a problem
  - an unambiguous computational procedure that takes some input and generates some output
  - a set of well-defined instructions for completing a task with a finite amount of effort in a finite amount of time
  - a set of instructions that can be mechanically performed in order to solve a problem

Key aspects of an algorithm

- An algorithm must be precise
  - The description of an algorithm must be clear and detailed enough so that someone (or something) can execute it
  - One way to ensure this is to describe it using actual computer code, which is guaranteed to be unambiguous
  - This is hard to read so pseudocode is often used instead, which is designed to be readable by humans
  - Since we assume no programming background, we will use English but will try hard to be clear and precise
- An algorithm operates on input and generates output
- An algorithm completes in a finite number of steps
  - This is a non-trivial requirement since certain methods may sometimes run forever!

Algorithms (as per Donald Knuth)

An algorithm should have the following characteristics:

1. **Finiteness:** It should terminate after a finite number of steps.
2. **Definiteness:** Each step of an algorithm must be precisely defined; the actions to be carried out must be rigorously and unambiguously specified for each case.
3. **Input:** An algorithm finitely many inputs, i.e., quantities given to it initially before the algorithm begins. (This includes the case of no input whatsoever.)
4. **Output:** An algorithm must produce finitely many (but at least one!) outputs, i.e., quantities having a specified relation to the inputs.
5. **Effectiveness:** All the operations to be performed in the algorithm must be sufficiently basic that they can in principle be done exactly and in a finite length of time by a person using pencil and paper.

Reference: Donald E. Knuth, *The Art of Computer Programming*. Volume 1, Chapter 1: Fundamental Algorithms.

Applications of algorithms

- Algorithms can implement many of the operations we study in this book, such as set membership and union
  - Data structures and the algorithms that operate on them are so important to CS that most CS majors are required to take a course on data structures
    - Sets and Sequences are examples of data structures
    - membership is a set operation implemented using an algorithm
    - union and intersection are also set operations implemented using an algorithm
    - How might you implement these operations?
  - Without such structures and without efficient algorithms for operating on them, you could never play a video game
- Algorithms can also used to implement mathematical processes/entities. Most mathematical functions are implemented using computer algorithms

Real world applications of algorithms

Algorithms are also used to solve specific, complex, real world problems:

- Google's success is largely due to its PageRank algorithm, which determines the "importance" of every web page
- Prim's algorithm can be used by a cable company to determine how to connect all of the homes in a town using the least amount of cable
- Dijkstra's algorithm can be used to find the shortest route between a city and all other cities
- The RSA encryption algorithm makes e-commerce possible by allowing for secure transactions over the Web

Algorithms and computers

- Algorithms have been used for thousands of years and have been executed by humans (possibly with pencil and paper)
  - We all know the algorithm for performing long division
- Work on algorithms exploded with the development of fast digital computers and are a cornerstone of Computer Science
  - Many algorithms are only feasible when implemented on computers
- But even with today's fast computers, some problems still cannot be solved using existing algorithms
  - The search for better and more efficient algorithms continues
- Interestingly enough, some problems have been shown to have no algorithmic solution (e.g., the "halting problem")

Searching and sorting algorithms

- Two of the most studied classes of algorithms in CS are searching and sorting algorithms
  - Search algorithms are important because quickly locating information is central to many tasks
  - Sorting algorithms are important because information can be located much more quickly if it is first sorted
- Searching and sorting algorithms are often used to introduce the topic of algorithms and we follow this convention

Search algorithms

- Problem: determine if an element x is in a list L
- We will look at two simple search algorithms
  - Linear search
  - Binary search
- The elements in L have some ordering, so that there is a first element, second element, etc.
- These algorithms can easily be applied to sets since we do not exploit this ordering (i.e., we do not assume the elements are sorted).

Linear search algorithm

The algorithm below will search for an element x in List L and will return "FOUND" if x is in the list and "NOT FOUND" otherwise. L has n items and $L[i]$ refers to the i^{th} element in L.

Linear Search Algorithm

1. **repeat** as i varies from 1 to n
2.   if $L[i] = x$ then return "FOUND" and stop
3. return "NOT FOUND"

Note: The repeat loop spans lines 1 and 2.

Efficiency of linear search algorithm

- If x appears once in L, on average how many comparisons (line 2) would you expect the algorithm to make ...
  - in the worst case?n comparisons
  - on average?$n/2$ comparisons
  - in the best case? 1 comparison
- If x does *not* appear in L, how many comparisons would you expect the algorithm to make?
  - n comparisons
- Would such an algorithm be useful for finding someone in a large (unsorted) phone book?
  - No, it would require scanning through the entire phone book (phone books are sorted for a reason)!
  - What if we had to check 1,000 people to see if they are in the phone book?
    - Then it would be even worse!

Binary search algorithm overview

- The binary search algorithm assumes that L is sorted
- This algorithm need not need explicitly examine each element
- At any given time it maintains a "window" in which element x may reside
  - The window is defined by the indices *min* and *max* which specify the leftmost and rightmost boundaries in L
- At each iteration of the algorithm the window is cut in half

Binary search algorithm

Binary Search Algorithm

1. Initialize $min \leftarrow 1$ and $max \leftarrow n$
2. **Repeat** until $min > max$
3. $midpoint = \frac{1}{2}(min + max)$
4. compare x to $L[midpoint]$
   (a) if $x = L[midpoint]$ then return "FOUND"
   (b) if $x > L[midpoint]$ then $min \leftarrow midpoint + 1$
   (c) if $x < L[midpoint]$ then $max \leftarrow midpoint - 1$
5. return "NOT FOUND"

Note: the repeat loop spans lines 2–4.

Binary search example

Use binary search to find the element "40" in the sorted list (10 30 40 50 60 70 80 90). List the values of *min*, *max* and *midpoint* after each iteration of step 4. How many values are compared to "40"?

1. $Min = 1$ and $max = 8$ and $midpoint = \frac{1}{2}(1 + 8) = 4$ (round down). Since $L[4] = 50$ and since $40 < 50$ we execute step 4c and $max = midpoint - 1 = 3$.
2. Now $min = 1$, $max = 3$ and $midpoint = \frac{1}{2}(1 + 3) = 2$. Since $L[2] = 30$ and $40 > 30$, we execute step 4b and $min = midpoint + 1 = 3$.
3. Now $min = 3$, $max = 3$ and $midpoint = \frac{1}{2}(3 + 3) = 3$. Since $L[3] = 40$ and $40 = 40$, we execute step 4a and return "FOUND."

During execution of the algorithm we check three values: 3, 4, and 5. Since we cut the list in half each iteration, it will shrink very quickly (the search will require about $\log_2 n$ comparisons).

Comparison of linear and binary search

- In the worst case, linear search will need to go through the entire list of n elements
- In the worst case, binary search will need to go through about $\log_2 n$ elements
- Binary search is much more efficient
  - If $n = 1K$ we have 1,024 vs. 10 comparisons
  - If $n = 1M$ we have ˜1,000,000 vs. 20 comparisons
  - If $n = 1G$ we have ˜1,000,000,000 vs 30 comparisons
- The drawback is that binary search requires sorting, and this requires a decent amount of work
  - But sorting only has to be done once and this will be worthwhile if we need to search the list many times

Sorting algorithms

- Sorting algorithms are one of the most heavily studied topics in Computer Science
- Sorting is critical if information is to be found efficiently (as we saw binary search exploits the fact that a list is sorted)
- There are many well known sorting algorithms in Computer Science
- We will study 2 sorting algorithms
  - BubbleSort: a very simple but inefficient sorting algorithm
  - MergeSort: a slightly more complex but efficient sorting algorithm

BubbleSort algorithm overview

- BubbleSort works by repeatedly scanning the list and in each iteration "bubbles" the largest element in the unsorted part of the list to the end
  - After 1 iteration largest element in last position
  - After 2 iterations largest element in last position and second largest element in second to last position
  - ...
- requires $n - 1$ iterations since at last iteration the only item left must already be in proper position (i.e., the smallest must be in the leftmost position)

BubbleSort algorithm

BubbleSort will sort the n-element list $L = (l_1, l_2, \ldots, l_n)$

BubbleSort Algorithm

1. **Repeat** as i varies from n down to 2
2. **Repeat** as j varies from 1 to $i - 1$
3. If $l_j > l_{j+1}$ swap l_j with l_{j+1}

- The outer loop controls how much of the list is checked each iteration. Only the unsorted part is checked. In the first iteration we check everything.
- The inner loop allows us to bubble up the largest element in the unsorted part of the list

BubbleSort example

Use BubbleSort to sort the list of number (9 2 8 4 1 3) into increasing order. Note that corresponds to Example 8.3 in the text.

Try it and compare your solution to the solution in the text.

- How many comparisons did you do each iteration?
- Can you find a pattern?
- This will be useful later when we analyze the performance of the algorithm.

MergeSort algorithm overview

- MergeSort is a *divide-and-conquer* algorithm
  - this means it divides the sorting problem into smaller problems
  - solves the smaller problems
  - then combines the solutions to the smaller problems to solve the original problem
- this deceptively simple algorithm is nonetheless much more efficient than the bubblesort algorithm
- It exploits the fact that combining two sorted lists is very easy
  - How would you sort (1 4 7 8) and (2 5 9)?
  - You would place your finger at the start of each list, copy over the smaller element under each finger, then advance that one finger.

MergeSort algorithm

MergeSort Algorithm

function mergeSort(L)

1. if L has one element then return(L); otherwise continue
2. $l_1 \leftarrow$ mergeSort(left half of L)
3. $l_2 \leftarrow$ mergeSort(right half of L)
4. $L \leftarrow$ merge(l_1, l_2)
5. return(L)

Description of MergeSort algorithm

- mergeSort is a recursive function
  - That means it calls itself
- If the input list contains one element it is trivially sorted so mergeSort is done
- Otherwise mergeSort calls itself on the left and right half of the list and then merges the two lists
- Each of these two calls to itself may lead to additional calls to itself
- Note that mergeSort will completely sort the left side of the original list before it actually starts sorting the right side

Example of MergeSort algorithm

How would mergesort sort the list (9 2 8 4 1 3) into increasing order?

To help show what is going on, the sorted lists that are about to be merged are shown in bold.

[9 2 8 4 1 3] → [9 2 8] [4 1 3] → [9 2] [8] [4 1 3]

[**9**] [**2**] [8] [4 1 3] → [**2 9**] [**8**] [4 1 3] → [2 8 9] [4 1 3]

[2 8 9] [4 1 3] → [2 8 9] [4 1] [3] → [2 8 9] [**4**] [**1**] [3]

[2 8 9] [**1 4**] [**3**] → [**2 8 9**] [**1 3 4**] → [1 2 3 4 8 9]

Analysis of algorithms

- We are done introducing our algorithms. Now we will analyze them.
- An algorithm is a set of instructions that solves a problem for all input instances
- But there may be many algorithms that can solve a problem and all of these are not equally good
- One criteria for evaluating an algorithm is *efficiency*
- The task of determining the efficiency of an algorithm is referred to as the *analysis of algorithms*
- Here we will learn to analyze only simple algorithms
  - There are entire courses on analysis of algorithms

How do we measure efficiency?

- When solving tasks, what are we most concerned with?
  - Hopefully most of us are concerned with correctness. But to be considered an algorithm the procedure must be correct (although a designer needs to make sure of this).
  - Most of us are pretty concerned with *time* and time is actually the main concern in evaluating the efficiency of algorithms
  - Space is also a concern, which, for algorithms, means what is the maximum amount of memory the algorithm will require at any one time
  - We will focus on time, although for some problems, space can actually be the main concern.

How do we measure the time of an algorithm?

- We could run the algorithm on a computer and measure the time it takes to complete
  - But what computer do we run it on? Different computers have different speeds.
  - We could pick one benchmark computer, but it would not stick around forever
  - Worse yet, the time taken by the algorithm is usually impacted by the specific input, so how do we handle that?

The run-time complexity of an algorithm

- The standard solution is to focus on the *run-time complexity* of an algorithm
- We determine how the number of operations involved in the algorithms grows relative to the length of the input
- Since inputs of the same length may still take different numbers of operations, we usually focus on the worst-case performance
  - We assume that the input is the hardest input possible

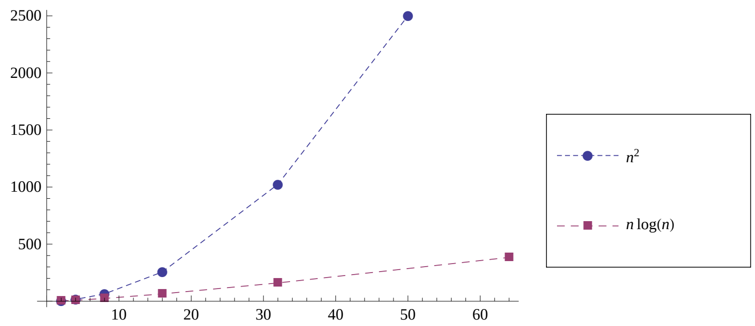
The running time of BubbleSort and MergeSort

- We can implement BubbleSort and MergeSort as a computer program
  - Then we can run them on various length lists and record the number of operations performed
  - Let bubblesortOps(n) and mergesortOps(n) represent the number of operations performed when the list has n elements
- The results might look like those below

n	2	4	8	16	32	64
bubblesortOps(n)	4	16	64	256	1024	4096
mergesortOps(n)	2	8	24	64	160	384

The running time of BubbleSort and MergeSort

- We can plot the data from the previous table to get a better visual picture of the growth rate for these functions



Run-time complexity of BubbleSort and MergeSort

- Using the data in the Table, can we determine closed formulas for bubblesortOps and mergesortOps?
  - We can see that $\text{bubblesortOps}(n) = n^2$
  - This is not easy to see, but $\text{mergesortOps}(n) = n \log_2 n$
- Normally one does not determine the run time complexity this way, but rather by analyzing the algorithm.
- We will show how to do this for some simple algorithms

Analysis of linear search algorithm

Linear Search Algorithm

1. **repeat** as i varies from 1 to n
2. if $L[i] = x$ then return "FOUND" and stop
3. return "NOT FOUND"

- How many operations will the linear search algorithm require?
- As stated earlier, since the algorithm checks at most n elements against x, the worst-case complexity requires n comparisons.
  - Note that this performance occurs only when x is not in the list or is the last element in the list.
- What is the best-case complexity of the algorithm?
  - 1, which occurs when x is the first item on the list

Average case complexity

- If you know that the element x to be matched is on the list, what is the average-case complexity of the algorithm?
  - The average case complexity of the algorithm should be $n/2$, since on average you should have to search half of the list
- At least for introductory courses on algorithms, the worst-case complexity is what is reported, since it is generally much easier to compute than the average case complexity.

Analysis of binary search algorithm

- The binary search algorithm, which assumes a sorted list, repeatedly cuts the list to be searched in half
  - If there is 1 element, it will require 1 comparison
  - If there are 2 elements, it may require 2 comparisons
  - If there are 4 elements, it may require 3 comparisons
  - If there are 8 elements, it may require 4 comparisons
  - In general, if there are n elements, how many comparisons will be required?
    - It will require $log_2 n$ comparisons
- If n is not a power of 2, you will need to round up the number of comparisons
  - Thus if there are 3 elements it may require 3 comparisons

Comparison of linear and binary search algorithms

- The linear search algorithm requires n comparisons worst case
- The binary search algorithm requires $log_2 n$ comparisons worst case
- Which one is faster? Is the difference significant?
- The binary search algorithm is much faster, in that it requires many fewer comparisons
  - If a list has 1 million elements then linear search requires 1,000,000 comparisons while binary search requires only about 20 comparisons!
- But the binary search algorithm requires that the list is sorted, whereas linear search does not.
- Since sorting requires $n \, log_2 n$ operations, which is more than n operations, it only makes sense to sort and then use binary search if many searches will be made
  - This is the case with dictionaries, phone books, etc.
  - This is not the case with airline reservation systems!

Analysis of the BubbleSort algorithm

- Analyzing the BubbleSort algorithm means determining the number of comparisons required to sort a list
- Recall that BubbleSort works by repeatedly bubbling up the largest element in the unsorted part of the list
- We can determine the number of comparisons by carefully analyzing the BubbleSort example we worked through earlier, when we sorted (9 2 8 4 1 3)
  - But we need to generalize from this example, so our analysis holds for all examples

Analysis of the BubbleSort algorithm (cont'd)

- If we apply BubbleSort to (9 2 8 4 1 3) how many comparisons do we do each iteration?
  - On iteration 1 we do 5 comparisons (6 unsorted numbers)
  - On iteration 2 we do 4 comparisons (5 unsorted numbers)
  - On iteration 3 we do 3 comparisons (4 unsorted numbers)
  - On iteration 4 we do 2 comparisons (3 unsorted numbers)
  - On iteration 5 we do 1 comparison (2 unsorted numbers)
  - On iteration 6 we do 0 comparisons (1 unsorted number)
- So how many total comparisons for a list with 6 items?
  - Number of comparisons $= 5 + 4 + 3 + 2 + 1 = 15$
- So how many comparisons for a list with n items?
  - $(n-1) + (n-2) + \cdots + 2 + 1$, or

$$\sum_{i=1}^{n-1} i$$

Analysis of the BubbleSort algorithm (cont'd)

- We want to know how the number of operations grows with n
- This is not obvious with the summation so we need to replace it with a closed formula
  - We can do this since it is known that

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

  - This was proven in the section on induction but is also based on the sum of n values equaling n times the average value
    - The average value of $1, 2, \ldots, n$ is $\frac{1}{2}(n+1)$
  - In this case, we are summing up to $n-1$ and not n, so substituting $n-1$ for n we get:
    - Number BubbleSort comparisons = $\frac{1}{2}(n-1)n = \frac{1}{2}(n^2 - n)$

Analysis of the BubbleSort algorithm (cont'd)

- So BubbleSort requires $\frac{1}{2}(n^2 - n)$ comparisons
- Computer scientists usually focus on the highest order term, so we say that the number of comparisons in bubblesort grows as n^2 or as the square of the length of the list
- BubbleSort can have problems if the list is very long

Analysis of the MergeSort algorithm

- Let a_n be the worst-case number of comparisons used by MergeSort algorithm when sorting a set of size n.
- MergeSort consists of the following steps when sorting an n-element set:
  - Sort left half, at cost $a_{n/2}$
  - Sort right half, at cost $a_{n/2}$
  - Merge the two halves, at cost at most $n-1$.
- So

$$a_n = \begin{cases} 2a_{n/2} + n - 1 & \text{if } n \text{ is even and } n \geq 2 \\ 0 & \text{if } n = 1 \end{cases}$$

- By induction, can prove

$$a_n = n \log_2 n - n + 1 \qquad \text{if } n \text{ is a power of 2.}$$

  (See next slide.)
- As we saw earlier, $n\, log_2 n$ grows much more slowly than n^2, so no one would ever use bubblesort unless the lengths of the lists are guaranteed to be small

Analysis of the MergeSort algorithm (cont'd)

Theorem

Let

$$a_n = \begin{cases} 2a_{n/2} + n - 1 & \textit{if } n \textit{ is even and } n \geq 2 \\ 0 & \textit{if } n = 1 \end{cases}$$

Then

$$a_n = n \log_2 n - n + 1 \qquad \textit{if } n \textit{ is a power of 2.}$$

Proof.

Let $n = 2^k$ and let $b_k = a_{2^k} = a_n$. We then have

$$b_k = \begin{cases} 2b_{k-1} + 2^k - 1 & \text{if } k \geq 2, \\ 0 & \text{if } k = 0 \end{cases}$$

and we want to show that

$$b_k = 2^k \cdot k - 2^k + 1 \qquad \text{for } k \in \mathbb{N}$$

Analysis of MergeSort (cont'd)

Proof (cont'd).

We're trying to show that if

$$b_k = \begin{cases} 2b_{k-1} + 2^k - 1 & \text{if } k \geq 1, \\ 0 & \text{if } k = 0 \end{cases}$$

then

$$b_k = 2^k \cdot k - 2^k + 1 \qquad \text{for } k \in \mathbb{N}$$

Use induction on k.

Base case: Let $k = 0$. Then

$$b_0 = 0$$
$$2^0 \cdot 0 - 2^0 + 1 = 0 - 1 + 1 = 0$$

and so the formula is okay when $k = 0$.

Analysis of MergeSort (cont'd)

Proof (cont'd).

Induction step: Let $m \in \mathbb{Z}^+$ and suppose that our formula is true when $k = m - 1$. Then

$$b_{m-1} = 2^{m-1} \cdot (m-1) - 2^{m-1} + 1$$

We then have

$$\begin{aligned} b_m &= 2b_{m-1} + 2^m - 1 \qquad \text{by recurrence relation} \\ &= 2\left(2^{m-1} \cdot (m-1) - 2^{m-1} + 1\right) + 2^m - 1 \\ &= 2^m \cdot m - 2^m + 1 = b_m, \end{aligned}$$

as required. □

Big-O notation

- Sometimes the exact formula for the operation count is too unwieldy:
  - The number of operations may depend on extraneous factors (programmer skill, compiler quality, system load, etc.).
  - It may be too difficult to obtain this formula.
  - The formula may be too complicated, thereby obscuring what's really going on.
- Real question: How fast does number of operations increase with input size n, as n gets large?
- We seek an *asymptotic upper bound*, given by O-notation ("Big-O notation").

Big-O notation (cont'd)

Main ideas is to get an upper bound on operation count, as follows:

- Hide low-order terms. So $n^2 + n = O(n^2)$.
- Hide multiplicative constants. So $3n^2 = O(n^2)$.

Some examples:

- $3n^2 + 2n = O(n^2)$.
- $2n^4 + n = O(n^4)$.
- $2^n + n^{42} = O(2^n)$.

But also note that $2n^4 + n = O(n^{10})$; the O-notation only provides an upper bound.

Big-O notation (cont'd)

- It's reasonable to drop lower-order terms.
- Why drop multiplicative constants?
  - We often care more about rate at which function grows than the exact amount of time algorithm takes for input of given size.
  - Example: Let $f_1(n) = 1000n^2$ and $f_2(n) = n^3$. Then $f_1(n) = O(n^2)$ and $f_2 = O(n^3)$. So f_1 is asymptotically better than f_2, even though $f_2(n) \le f_1(n)$ when $n \le 1000$.
- Asymptotic complexity of some algorithms we've considered:

Algorithm	Time Complexity
linear search	$O(n)$
binary search	$O(\log_2 n)$
insertion sort	$O(n^2)$
bubblesort	$O(n^2)$
mergesort	$O(n \log_2 n)$

Big-O notation (cont'd)

Definition

Let $f, g\ \mathbb{N} \to \mathbb{R}^{\ge 0}$. We say that $f(n) = O\big(g(n)\big)$ (as $n \to \infty$) iff there exist constants $c \ge 0$ and $n_0 \in \mathbb{N}$ such that

$$0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 .$$

Example

Let $f(n) = 2n^4 + 7n^3 + 100n^2 \log_2 n$. Prove that $f(n) = O(n^4)$.

Solution

Note that for $n \ge 1$, we have

$$0 \le f(n) \le 2n^4 + 7n^4 + 100n^4 = 114n^4.$$

So the O-condition holds with $n_0 = 1$ and $c = 114$.