CISC 1400
Discrete Structures
Chapter 2
Sequences

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Outline

- Finding patterns
- Notation
  - Closed form
  - Recursive form
  - Converting between them
- Summations

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15, 21
- 1, 2, 6, 24, 120, 720
- 1, 1, 2, 3, 5, 8, 13, 21

Discovering the pattern

- Each term might be related to previous terms
- Each term might depend on its position number (1st, 2nd, 3rd, . . .)
- “Well-known” sequences (even numbers, odd numbers)
- Some (or all) of the above

$2, 4, 6, 8, 10, \ldots$

Can we relate a term to previous terms?

- Second term is 2 more than first term.
- Third term is 2 more than second term.

  $\vdots$
- Any given term is 2 more than previous term.

$2, 4, 6, 8, 10, \ldots$

Can we describe each term by its position in the sequence?

- Term at position 1 is 2.
- Term at position 2 is 4.
- Term at position 3 is 6.

  $\vdots$
- Term at position n is $2n$.

Mathematical notation

- Write term in a sequence as a lower case letter, followed by a *subscript* denoting position number of the term (e.g., a_1, b_7, z_k).
- For the sequence $2, 4, 6, 8, 10, \ldots$:
  - $a_1 = 2$.
  - $a_2 = 4$.
  - a_n is nth term in the sequence.
- What is a_3? 6
- What is a_5? 10
- What is a_6? 12
- What is a_n if $n = 5$? 10
- What is a_{n+1} if $n = 5$? 12

Recursive formula

- *Recursive formula* for a sequence: each term is described in relation to previous term(s). For example:

  $$a_n = 2a_{n-1}$$

  So

  $$\begin{aligned} a_3 &= 2a_2 = 2 \cdot (2a_1) = 4a_1 = 4 \cdot (2a_0) = 8a_0 \\ &= 8 \cdot (2a_{-1}) = 16a_{-1} = \ldots \end{aligned}$$

- Problem: Need a starting point (initial condition) such as

  $$a_1 = 1$$

- So let's try

  $$\begin{aligned} a_n &= 2a_{n-1} \qquad \text{for } n \geq 2 \\ a_1 &= 1 \end{aligned}$$

- Example:

  $$a_3 = 2a_2 = 2 \cdot (2a_1) = 4a_1 = 4 \cdot 1 = 4$$

Fibonacci sequence

- $1, 1, 2, 3, 5, 8, 13, \ldots$
- Recursive formula:

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \\ a_2 &= 1 \\ a_1 &= 1 \end{aligned}$$

- What's a_{10}? Top-down solution:

$$a_{10} = a_9 + a_8 = (a_8 + a_7) + (a_7 + a_6) = a_8 + 2a_7 + a_6 \ldots$$

  Too hard!
- Better way? Work bottom-up via a grid.

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	3	5	8	13	21	34	55

Recursion

- Recursive formula corresponds to "recursive function" in a programming language.
- Fibonacci formula

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \\ a_2 &= 1 \\ a_1 &= 1 \end{aligned}$$

- Recursive function

```
def fib(n):
    if n==1 or n==2:
        return 1
    else:
        return fib(n-1) + fib(n-2)
```


Exercise: Find recursive formula

- $2, 4, 6, 8, 10, \ldots$

$$\begin{aligned} a_n &= a_{n-1} + 2 \qquad \text{for } n \geq 2 \\ a_1 &= 2 \end{aligned}$$

- $1, 3, 6, 10, 15, \ldots$

$$\begin{aligned} a_n &= a_{n-1} + n \qquad \text{for } n \geq 2 \\ a_1 &= 1 \end{aligned}$$

- $2, 2, 4, 6, 10, 16, \ldots$

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \\ a_2 &= 2 \\ a_1 &= 2 \end{aligned}$$

Finding a closed formula

- Write each term in relation to its position
- Example: $2, 4, 6, 8, 10, \ldots$
  - $a_1 = 2 = 2 \cdot 1$
  - $a_2 = 4 = 2 \cdot 2$
  - $a_3 = 6 = 2 \cdot 3$
  - More generally, $a_n = 2n$.

Find the closed formulas

- $1, 3, 5, 7, 9, \dots a_n = 2n - 1$
- $3, 6, 9, 12, 15, \dots b_n = 3n$
- $8, 13, 18, 23, 28, \dots c_n = 5n + 3$
- $3, 9, 27, 81, 243, \dots d_n = 3^n$

Recursive formulas vs. closed formulas

- Recursive formula
  - It's often easier to find a recursive formula for a given sequence.
  - It's often harder to evaluate a given term.
- Closed formula
  - It's often harder to find a closed formula for a given sequence.
  - It's often easier to evaluate a given term.

Closed formula ⇒ recursive formula

- Write out a few terms.
- See if you can figure out how a given term relates to previous terms.
- Example: $r_n = 3n + 4$.

n	1	2	3	4	5	...
r_n	7	10	13	16	19	...

We find

$$r_n = r_{n-1} + 3 \qquad \text{for } n \geq 2$$
$$r_1 = 7$$

Closed formula ⇒ recursive formula

Can also use algebraic manipulation. Let's try

$$r_n = 3n + 4$$

again.

- Initial condition is easiest—substitute $n = 1$ into closed form:
$$r_1 = 3 \cdot 1 + 4 = 7$$
- Recursive formula: Try to describe r_n in terms of r_{n-1}:
$$r_n = 3n + 4$$
$$r_{n-1} = 3(n-1) + 4 = 3n - 3 + 4 = 3n + 1$$
So
$$r_n - r_{n-1} = (3n + 4) - (3n + 1) = 3,$$
i.e.,
$$r_n = r_{n-1} + 3$$

Another example

$$s_n = 2^n - 2$$

- Initial condition: $s_1 = 2^1 - 2 = 0$.
- Recursive formula: We have

  $$s_n = 2^n - 2$$

  and

  $$s_{n-1} = 2^{n-1} - 2$$

  So

  $$\begin{aligned} s_n &= 2^n - 2 = 2 \cdot 2^{n-1} - 2 = 2 \cdot 2^{n-1} - 4 + 2 \\ &= 2 \cdot (2^{n-1} - 2) + 2 \\ &= 2s_{n-1} + 2 \end{aligned}$$

Exercise

Find the recursive formulas for the following sequences:

- $a_n = 2n + 7$
  - $a_1 = 9$
  - $a_n = a_{n-1} + 2$ for $n \geq 2$.
- $b_n = 2^n - 1$
  - $b_1 = 1$
  - $b_n = 2b_{n-1} + 1$ for $n \geq 2$.

Summations

Summing the terms in a sequence: important enough to have its own notation ("sigma notation"):

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Parts of speech?

- Large Σ: "summation"
- $i = 1$ at bottom: We want to start summation at term #1 of the sequence.
- n at the top: We want to stop summation at the nth term of the sequence
- Portion to the right of the $\Sigma_{i=1}^{n}$: Closed form of sequence we want to sum.

Examples of Σ-notation:

$$\begin{aligned} \sum_{i=1}^{5}(3i+7) &= (3 \cdot 1 + 7) + (3 \cdot 2 + 7) + (3 \cdot 3 + 7) + (3 \cdot 4 + 7) \\ &\quad + (3 \cdot 5 + 7) \\ &= 10 + 13 + 16 + 19 + 22 = 80 \end{aligned}$$

$$\begin{aligned} \sum_{j=2}^{6}(j^2 - 2) &= (2^2 - 2) + (3^2 - 2) + (4^2 - 2) + (5^2 - 2) + (6^2 - 2) \\ &= 2 + 7 + 14 + 23 + 34 = 80 \end{aligned}$$

Note: Parentheses are important!

$$\sum_{i=1}^{5} 3i + 7 = (3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 3 \cdot 5) + 7 = 52$$

Converting a sum into Σ-notation

$$3+7+11+15+19 = \sum_{i=1}^{5}(4i-1)$$
$$= \sum_{j=1}^{5}(4j-1)$$
$$0+3+8+15+24 = \sum_{k=1}^{5}(k^2-1)$$

Mathematical induction

Suppose you have a statement $P(n)$ about the positive integer n. How would you prove that $P(n)$ is true for *all* $n \in \mathbb{Z}^+$?

Prove $P(1)$

Prove $P(2)$

Prove $P(3)$

Prove $P(4)$

$\vdots$

Prove $P(100000000)$

But this doesn't guarantee that $P(n)$ is true for all n; maybe $P(100000001)$ is false!!

Example: Sum of the first n positive integers

Want to show that

$$\sum_{j=1}^{n} j = \tfrac{1}{2}n(n+1) \qquad \forall\, n \in \mathbb{Z}^+,$$

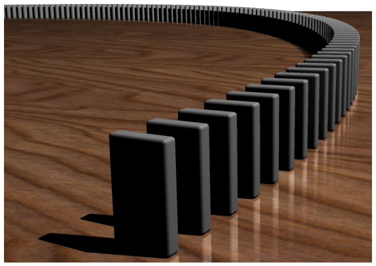
or, if you prefer,

$$1+2+\cdots+n = \tfrac{1}{2}n(n+1) \qquad \forall\, n \in \mathbb{Z}^+.$$

"How on earth did you come up with this formula in the first place?" Later . . .

n	1	2	3	4	5	6	7	8	9	10
$\sum_{j=1}^{n} j$	1	3	6	10	15	21	28	36	45	55
$\frac{1}{2}n(n+1)$	1	3	6	10	15	21	28	36	45	55

Dominoes!!



Suppose:

- You're going to push the first one over.
- If any given domino has fallen down, the next one after it will also fall down.

They'll all fall down!

Theorem (First Principle of Mathematical Induction)

Let $P(n)$ be a statement about the positive integer $n \in \mathbb{Z}^+$.
Suppose we can prove the following:

- ***Basis step:*** *$P(1)$ is true.*
- ***Induction step:*** *If $P(k)$ is true for some arbitrary $k \in \mathbb{Z}^+$, then $P(k+1)$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Why?

$P(1)$ is true (basis step).
$P(1)$ being true implies $P(1+1) = P(2)$ is true (induction step).
$P(2)$ being true implies $P(2+1) = P(3)$ is true (induction step).
$P(3)$ being true implies $P(3+1) = P(4)$ is true (induction step).
$P(4)$ being true implies $P(4+1) = P(5)$ is true (induction step).
... and so on

Example: Sum of the first n positive integers (cont'd)

Theorem

$$\sum_{j=1}^{n} j = \tfrac{1}{2}n(n+1) \qquad \forall\, n \in \mathbb{Z}^+$$

Proof (by induction): For $n \in \mathbb{Z}^+$, the statement $P(n)$ we're trying to prove is

$$\sum_{j=1}^{n} j = \tfrac{1}{2}n(n+1). \tag{1}$$

Basis step: Let $n = 1$. Then

$$\sum_{j=1}^{n} j = \sum_{j=1}^{1} j = 1 \qquad \text{and} \qquad \tfrac{1}{2}n(n+1) = \tfrac{1}{2}\cdot 1\cdot(1+1) = 1.$$

So formula (1) is true when $n = 1$, i.e., $P(1)$ is true.

Induction step: Let $k \in \mathbb{Z}^+$, and suppose that $P(k)$ is true; we need to show that $P(k+1)$ is true.
Since $P(k)$ is true, we know that

$$\sum_{j=1}^{k} j = \tfrac{1}{2}k(k+1)$$

Using this as a starting point, we want to show that $P(k+1)$ is true, i.e., that

$$\sum_{j=1}^{k+1} j = \tfrac{1}{2}(k+1)\big((k+1)+1\big) = \tfrac{1}{2}(k+1)(k+2).$$

Induction step (cont'd): But

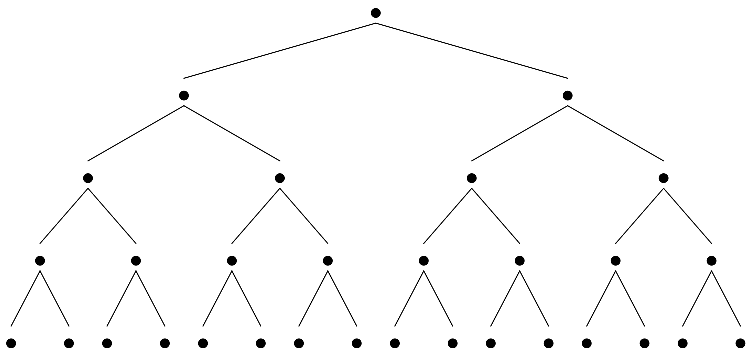
$$\begin{aligned}\sum_{j=1}^{k+1} j &= \left(\sum_{j=1}^{k} j\right) + (k+1)\\ &= \tfrac{1}{2}k(k+1) + (k+1) \qquad \text{by the induction hypothesis}\\ &= \left(\tfrac{1}{2}k+1\right)(k+1)\\ &= \tfrac{1}{2}(k+2)(k+1),\end{aligned}$$

as required to prove that $P(k+1)$ is true.

Since we have proved the basis step and the induction step, it follows that $P(n)$ is true for all $n \in \mathbb{Z}^+$. □

Example: Number of leaves in complete binary tree

Here's a complete binary tree with five levels:



- Terminology:
  - Tree? All edges go from a given level to the next level.
  - Binary? No more than two descendants per node.
  - Complete? Each node has exactly two descendants.
- **Question:** How many nodes branch out from the nth level of a complete binary tree?
- Get an idea by making a table. Let b_n denote the number of nodes branching out from the nth level. Looking at the drawing we saw earlier:

n	1	2	3	4	5
b_n	2	4	8	16	32

- This suggests that $b_n = 2^n$.

Theorem

For $n \in \mathbb{Z}^+$, let b_n be the number of nodes branching out from the nth level of a complete binary tree. Then $b_n = 2^n$

Proof (by induction): For $n \in \mathbb{Z}^+$, the statement $P(n)$ we're trying to prove is

$$b_n = 2^n. \tag{2}$$

Basis step: Let $n = 1$. Looking at the first level of the binary tree, it is immediately clear that $b_1 = 2$. So $P(1)$ is true.

Induction step: Let $k \in \mathbb{Z}^+$, and suppose that $P(k)$ is true; we need to show that $P(k+1)$ is true.
Since $P(k)$ is true, we know that $b_k = 2^k$.

- Since we're working with a complete binary tree, each node at any level branches out to two nodes at the next level.
- Each node at level k branches out to two nodes at level $k+1$.
- So $b_{k+1} = 2b_k$.

Hence

$$\begin{aligned} b_{k+1} &= 2b_k \\ &= 2 \cdot 2^k \qquad \text{(by the induction hypothesis)} \\ &= 2^{k+1}, \end{aligned}$$

as required to prove that $P(k+1)$ is true.

Since we have proved the basis step and the induction step, it follows that $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$