CISC 1400
Discrete Structures
Chapter 6
Counting

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Why talk about counting in a college-level course?

- Counting isn't as easy as it looks.
  - Simple sets: trivial to count.
  - Complicated sets: hard to count.
    - Facebook FOAF.
    - Number of ways to fill a committee.
    - Number of ways to fill a slate of officers.
    - Number of outcomes in a game (chess, poker, . . .).
  - Methodically enumerating a set.
- Connection between counting and probability theory.

Outline

- Counting and how to count
- Elementary rules for counting
  - The addition rule
  - The multiplication rule
  - Using the elementary rules for counting together
- Permutations and combinations
- Additional examples

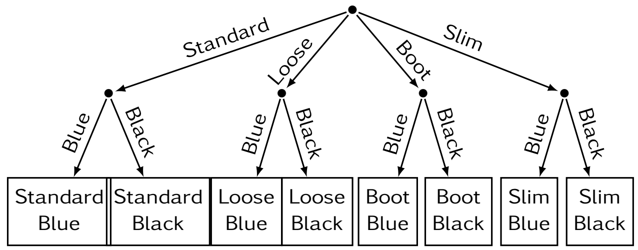
Counting and how to count

- Some things are easy to count (e.g., number of students in this class).
- Some things are harder to count.
- **Example:** You are asked to select a pair of men's jeans.
  - Four styles are available (standard fit, loose fit, boot fit, and slim fit).
  - Each style comes in two colors (blue or black).
- You *could* list all possibilities for this problem.

	Jeans Style			
Color	Standard	Loose	Boot	Slim
Blue	Standard-Blue	Loose-Blue	Boot-Blue	Slim-Blue
Black	Standard-Black	Loose-Black	Boot-Black	Slim-Black

Counting and how to count (cont'd)

- This doesn't generalize.
  - What if more than two "features"?
- One idea: Use a *tree structure* to help you enumerate the choices.



Counting and how to count (cont'd)

Example: We toss a penny, a nickel, and a dime into the air. How many different configurations?

- How to encode? As a triple:

  (penny's state, nickel's state, dime's state)

- Configurations?

$$C = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}.$$

- How many configurations? 8.
- How to count configurations without listing?

Elementary rules of counting

- Two basic rules:
  - Addition rule
  - Multiplication rule
- Using these rules together

Elementary rules of counting: the addition rule

- **Example:** You need to purchase one shirt of any kind. The store has five short sleeve shirts and eight long sleeve shirts. How many possible ways are there to choose a shirt?
- **Solution:** $8 + 5 = 13$.
- **Addition rule:**
  - If we have two choices C_1 and C_2, with C_1 having a set O_1 of possible outcomes and C_2 having a set O_2 of possible outcomes, with $|O_1| = n_1$ and $|O_2| = n_2$, then the total number of outcomes for C_1 *or* C_2 occurring is $n_1 + n_2$.
  - If we have k choices $C_1, \ldots, C_k$ having $n_1, \ldots, n_k$ possible outcomes, then the total number of ways of C_1 occurring *or* C_2 occurring *or* … *or* C_k occurring is $n_1 + n_2 + \cdots + n_k$.
- Fairly straightforward.

Elementary rules of counting: the multiplication rule

- In our jeans example,

$$\text{\# of jeans configurations} = (\text{\# number of styles}) \times (\text{\# of colors})$$

- **Multiplication rule:**
  - If we have two choices C_1 and C_2, with C_1 having a set O_1 of possible outcomes and C_2 having a set O_2 of possible outcomes, with $|O_1| = n_1$ and $|O_2| = n_2$, then the total number of possible outcomes for C_1 *and* C_2 occurring is $n_1 \times n_2$.
  - More generally, if we have k choices $C_1, \dots, C_k$ having $n_1, \dots, n_k$ possible outcomes, then the total number of ways of C_1 occurring *and* C_2 occurring *and* … *and* C_k occurring is $n_1 \times n_2 \times \cdots \times n_k$.
- Roughly speaking:
  - addition rule: "or" rule
  - multiplication rule: "and" rule

Elementary rules of counting: the multiplication rule (cont'd)

Example: Solve jeans problem via multiplication rule …

- four styles (standard, loose, slim, and boot fits) and
- two colors (black, blue)

Solution: Our choices?

$$C_1 = \text{"choose the jeans style"},$$
$$C_2 = \text{"choose the jeans color"}.$$

Our outcomes?

$$O_1 = \{\text{standard fit, loose fit, boot fit, slim fit}\},$$
$$O_2 = \{\text{black, blue}\}.$$

Now determine the cardinalities of the sets:

$$n_1 = |O_1| = 4 \qquad n_2 = |C_2| = 2.$$

Now we apply the multiplication rule

$$\text{Total number of outcomes} = n_1 \times n_2 = 4 \times 2 = 8.$$

Elementary rules of counting: the multiplication rule (cont'd)

- Why does the multiplication rule work?
- The set of possible outcomes is for O_1 *and* O_2 occurring is $O_1 \times O_2$.
- We know that $|O_1 \times O_2| = |O_1| \cdot |O_2|$.
- This is the multiplication rule!

Elementary rules of counting: the multiplication rule (cont'd)

Example: Suppose that you flip a coin twice and record the outcome (head or tail) for each flip. How many possible outcomes are there?
Solution: There are two choices, C_1 and C_2, corresponding to the two coin flips. C_1 *and* C_2 must occur, so the multiplication rule applies. Each choice has two possible outcomes, thus $n_1 = 2$ and $n_2 = 2$. Thus by the multiplication principle of counting, there are $2 \times 2 = 4$ possible outcomes.

Elementary rules of counting: the multiplication rule (cont'd)

Example: You are asked to flip a coin five times and to record the outcome (head or tail) for each flip. How many possible outcomes are there?

Solution:

- This example differs from the previous one only in that there are five choices instead of two.
- For each choice there are two possible outcomes.
- The total number of outcomes is

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. If a number may be selected more than once, then how many ways can you fill out the lottery card?

Solution:

- There are five choices, corresponding to the five numbers that you must choose.
- Each of the five choices must occur, so the multiplication rule applies.
- Each choice has twenty possible outcomes (i.e., you pick a number between 1 and 20).
- There are

$$20 \times 20 \times 20 \times 20 \times 20 = 20^5 = 3{,}200{,}000$$

possible ways to fill out the lottery card.

Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. The numbers are chosen by the lottery commission from a bin and once a number is chosen it is discarded and cannot be chosen again. In how many ways can you fill out the lottery card?

Elementary rules of counting: the multiplication rule (cont'd)

Solution:

- Close to the previous one, but a number cannot be chosen more than once.
- Hence, the number of possible outcomes for each choice is progressively reduced by one.
- Number the five choices $C_1 \ldots C_5$ such that C_1 corresponds to the first number selected and C_5 to the last number selected.
- The number of outcomes for C_1 is 20, for C_2 is 19, for C_3 is 18, for C_4 is 17 and for C_5 is 16.
- Thus the number of possible outcomes is

$$20 \times 19 \times 18 \times 17 \times 16 = 1{,}860{,}480.$$

Elementary rules of counting: the multiplication rule (cont'd)

- Don't be misled by the word "and"!
- **Example:** How many ways are there to choose one class among 5 day classes and 2 evening classes?
- **Solution:** $5 + 2 = 7$ ways.

Elementary rules of counting: combining the rules together

- **Example:** How many odd three-digit numbers are there (allowing leading zeros, such as 007)?
- **First solution:**
  - We have three choices, one per digit. Let C_1, C_2, C_3 denote the choices for the first, second, third digits.
  - $O_1 = O_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, while $O_3 = \{1, 3, 5, 7, 9\}$.
  - So $|O_1| = 10$, $|O_2| = 10$, $|O_3| = 5$.
  - Hence there are $10 \times 10 \times 5 = 500$ outcomes.

Elementary rules of counting: combining the rules together

- **Example:** How many odd three-digit numbers are there (allowing leading zeros, such as 007)?
- **Second solution:**
  - Number of outcomes = number of outcomes where the three-digit number ends in a 1 *or* 3 *or* 5 *or* 7 *or* 9.
  - Each of these five cases has $10 \times 10 = 100$ outcomes.
  - So there are $5 \times 100 = 500$ outcomes overall.

Facts about playing cards

- A deck of cards contains 52 cards.
- Each card belongs to one of four *suits*

  ♣ (Clubs), ♢ (Diamonds), ♡ (Hearts), ♠ (Spades)

  and one of thirteen denominations

  2, 3, 4, 5, 6, 7, 8, 9, 10, J(ack), Q(ueen), K(ing), A(ce).
- The clubs and spades are black and the diamonds and hearts are red.
- Unless otherwise specified, assume that for any example you begin with a complete deck and that as cards are dealt they are not immediately replaced back into the deck.
- We abbreviate a card using the denomination and then suit, such that 2♡ (or 2H) represents the 2 of Hearts.

Poker hands

- In standard poker you receive 5 cards.
- The suits are equally important.
- The face values are ordered

  $$2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < J < Q < K < A$$

- While you can later discard cards and then replace them, for most of our examples we will only consider the initial configuration.
- *Pair* (*two of a kind*): two cards that are the same denomination, such as a pair of 4's.
- *Three of a kind* and *four of a kind* are defined similarly.
- *Full house*: three of one kind and a pair of another kind.
- *Straight*: the cards are in sequential order, with no gaps.
- *Flush*: all five cards are of the same suit.
- *Straight flush*: all five cards are of the same suit and in sequential order (i.e., a straight *and* a flush).

Poker hands (cont'd)

Ordering of the hands (highest to lowest):

- straight flush (with a "royal flush" [ace high] the highest possible hand of all)
- four of a kind
- full house
- flush
- straight
- three of a kind
- two pairs
- one pair
- high card

A poker example

In how many ways can you draw a flush in poker, assuming that the order of the five cards drawn matters? (We will learn how to relax this assumption in the next section.)

- There are four basic ways to get a flush: all clubs or all diamonds or all hearts or all spades.
- Each is an outcome satisfying the condition of drawing a flush; we want to determine the total number of outcomes of these four *non-overlapping* outcomes.
- How many ways can we get an all-clubs flush? By multiplication rule to select 5 cards without replacement,

  $$\text{\# ways to draw five clubs} = 13 \times 12 \times 11 \times 10 \times 9 = 154{,}440.$$

- Therefore, by the addition rule, there are $4 \times 154{,}440 = 617{,}760$ ways to get a flush.

Permutations and Combinations

- Sometimes order matters, sometimes it doesn't.
- **Example:** How many ways to get a royal flush in spades?

  $$A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit$$

  - If order matters, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.
  - If order does not matter, there is only 1 way.
- Order matters: *permutation*
- Order doesn't matter: *combination*

Permutations

- *Permutation:* order matters, cannot reuse objects.
- Phone numbers 123-456-7890 and 789-012-3456 are different. These are two permutations of the set of digits.
- **Example:** How many ways to seat 5 children in 5 chairs?
  - Both criteria for permutations are satisfied.
  - Counting permutations of 5 children.
  - By multiplication rule, there are

    $$5 \times 4 \times 3 \times 2 \times 1 = 120$$

    different seating arrangements.
- **Example:** How many ways to seat 10 children in 5 chairs?
  - Both criteria for permutations are satisfied.
  - Counting permutations of 10 children, *chosen 5 at a time.*
  - By multiplication rule, there are

    $$10 \times 9 \times 8 \times 7 \times 6 = 30{,}240$$

    different seating arrangements.

Permutations (cont'd)

- **Notation:** $P(n, r)$ is the number of permutations of n objects, chosen r at a time.
- Formula for $P(n, r)$?

  $$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$

- Excursus on factorials
  - $n!$ is the product of the natural numbers $1, 2, \ldots, n$.
  - Semi-special case: $0! = 1$.
  - Table of factorials:

n	0	1	2	3	4	5	6	7	...
$n!$	1	1	2	6	24	120	720	5,040	...

n	...	8	9	10	...
$n!$	...	40,320	362,880	3,628,800	...

- "Simplified" formula for $P(n, r)$?

  $$P(n, r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Permutations (cont'd)

- **Example (cont'd):** We have

  $$P(10, 5) = 10 \times 9 \times 8 \times 7 \times 6 = 30{,}240.$$

- We also have

  $$P(10, 5) = \frac{10!}{(10-5)!} = \frac{10!}{5!}.$$

- Save some work: cancel common factors

  $$\begin{aligned} P(10, 5) &= \frac{10!}{(10-5)!} = \frac{10!}{5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times \not{5} \times \not{4} \times \not{3} \times \not{2} \times \not{1}}{\not{5} \times \not{4} \times \not{3} \times \not{2} \times \not{1}} \\ &= 10 \times 9 \times 8 \times 7 \times 6 = 30{,}240. \end{aligned}$$

- All our answers agree.

Permutations (cont'd)

- Sanity check:
  - $P(n, r)$ counts something.
  - Thus $P(n, r)$ *must* be a non-negative integer.
  - The formula

    $$P(n, r) = \frac{n!}{(n-r)!}$$

    appears to involve division.
  - You will *always* be able to use the cancellation trick to get rid of divisions.
  - Alternatively, use the formula

    $$P(n, r) = n(n-1)(n-2)\ldots(n-r+1).$$

    (There are r factors.)
- If the answer you get to a permutation problem is anything other than a non-negative integer, *go back and check your work*!

Permutations (cont'd)

- **Example:** In how many ways can we choose a 3-person slate of officers (president, vice-president, secretary) out of the 10 members in this class?
- **Solution:** We need to choose 3 distinct people out of 10, with order mattering.
- So
$$P(10,3) = 10 \times 9 \times 8 = 720.$$

Permutations (cont'd)

- **Example:** In major league baseball, each team has a 25-player roster. How many possible batting orders are there for such a roster?
- **Solution:** Check that this is a permutation.
- Total number of batting orders is
$$P(25,9) = \frac{25!}{16!} = 25 \times 24 \times \cdots \times 17 = 741{,}354{,}768{,}000.$$

Permutations (cont'd)

- **Example:** Let r be a relation on a finite set S. How many triples of S-members do we need to examine to determine whether or not S is transitive?
- **Solution:** We need to check all triples of distinct S members. Order matters. So we use a permutation.
- If $n = |S|$, the total number of triples to check is
$$P(n,3) = n(n-1)(n-2) = n^3 - 3n^2 + 2n \asymp n^3.$$
- Check some values

n	10	100	1,000	10,000
$P(n,3)$	720	970,200	9.9702×10^8	9.997×10^{11}
n^3	1,000	1,000,000	10^9	10^{12}
$P(n,3)/n^3$	0.72	0.9702	0.9970	0.9997

Combinations

- For some problems, order matters. (Baseball lineup problem.)
- For some problems, order does *not* matter.
- **Example:** We need to choose a 12-person jury from a pool of 1000 people. The order does not matter here. We want the number of *combinations* of 1000 persons, chosen 12 at a time.
- **Notation:** $C(n,r)$ denotes the number of *combinations* of n objects, chosen r at a time. Here the order *does not* matter, and we are not allowed to reuse objects. We often read this as "n choose r".
- Formula for combinations:
$$C(n,r) = \frac{n!}{(n-r)!r!}$$
Why?
$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!}$$

Combinations (cont'd)

- **Example:** In how many ways can we choose a 3-person committee out of a 10-member class?
- **Solution:** We need to choose 3 distinct people out of 10, with order not mattering.
- So

$$\begin{aligned} C(10,3) &= \frac{10!}{3!\cdot 7!} \\ &= \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(3\times 2\times 1)(7\times 6\times 5\times 4\times 3\times 2\times 1)} \\ &= \frac{10\times \cancel{9}^{3}\times \cancel{8}^{4}\times \cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{(\cancel{3}\times \cancel{2}\times \cancel{1})(\cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1})} \\ &= 10\times 3\times 4 = 120. \end{aligned}$$

Combinations (cont'd)

- Sanity check:
  - $C(n,r)$ counts something.
  - Thus $C(n,r)$ *must* be a non-negative integer.
  - The formula
    $$C(n,r) = \frac{n!}{(n-r)!r!}$$
    involves division.
  - You will *always* be able to use the cancellation trick to get rid of divisions.
- If the answer you get to a combination problem is anything other than a non-negative integer, *go back and check your work*!

Additional Examples

Example: A typical telephone number has 10 digits (e.g., 555-817-4495), where the first three are known as the area code and the next three as the exchange.

1. Assuming *no* restrictions, how many possible (three-digit) area codes are there?
   Solution: $10\times 10\times 10 = 1{,}000$ three-digit area codes.
2. Assuming that the middle digit of the area code must be a 0 or a 1 (which was required until recently), how many possible (3 digits) area codes are there?
   Solution: $10\times 2\times 10 = 200$ area codes.
3. Assuming no restrictions whatsoever, how many possible values are there for the full 10-digit phone number?
   Solution: $10^{10} = 10{,}000{,}000{,}000$ phone numbers
4. If the only restriction is that no digit may be used more than once, how many possible 10-digit phone numbers are there?
   Solution: $10\times 9\times\cdots\times 1 = 10! = 3{,}628{,}800$ phone numbers.

Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt "two pairs"?

Solution:

- There are $C(13,2)$ ways to identify the two denominations.
- For each denomination, there are $C(4,2)$ ways to choose two of the four cards. Do this twice.
- Pick the last card? 11 ways for each of 4 suits.
- Final answer:

$$C(13,2)\times C(4,2)\times C(4,2)\times 11\times 4 = 123{,}552 \text{ ways.}$$

Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt "three of a kind"?

Solution:

- We can choose the denomination with 3 of a kind in 13 ways.
- There are $C(4,3)$ ways to choose the three cards of said denomination.
- The two remaining cards must come from the other 12 denominations. They can't be the same, since this would yield a full house. Since there are 4 suits, there are $C(12,2) \times 4 \times 4$ ways of choosing these two cards.
- Final answer:

$$13 \times C(4,3) \times C(12,2) \times 4 \times 4 = 54{,}912 \text{ ways.}$$

Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt a "full house"?

Solution:

- A full house requires 3 of a kind and also 2 of a different kind.
- We can choose the denomination with 3 of a kind in 13 ways and then we can choose the 3 specific cards in $C(4,3)$ ways.
- Then we can choose the denomination with the 2 of a kind in 12 ways and choose the 2 specific cards in $C(4,2)$ ways.
- Final answer:

$$13 \times C(4,3) \times 12 \times C(4,2) = 3{,}744 \text{ ways.}$$

Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word `MISSISSIPPI`?

Solution #1:

- Multiplication rule: 11! ways.
- Since there are
  - 4 instances of `S` and `I`
  - 2 instances of `P`

  not all 11! ways are distinguishable.
- Since there are 4 instances of `S`, their appearance can be permuted in 4! different ways. So we need to divide the current answer by 4!, getting 11!/4!.
- Since there are 4 instances of `I`, their appearance can be permuted in 4! different ways. So we need to divide the current answer by 4!, getting 11!/(4!4!).

Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word `MISSISSIPPI`?

Solution #1 (contd):

- Answer so far (accounting for multiple `S` and `I`): 11!/(4!4!).
- Since there are 2 instances of `P`, their appearance can be permuted in 2! different ways. So we need to divide the current answer by 2!, getting 11!/(4!4!2!).
- Final answer:

$$\frac{11!}{4!4!2!} = 11 \times 10 \times 9 \times 7 \times 5 = 34{,}650 \text{ ways.}$$

Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word MISSISSIPPI?

Solution #2: Use a "fill-in-the-blank" approach, starting with 11 blanks

___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

- Can assign the one M in $C(11,1) = 11!/(10! \times 1!) = 11$ ways.
- Can assign the two P's in $C(10,2)$ ways.
- Can assign the four S's in $C(8,4)$ ways.
- Can assign the four I's in $C(4,4) = 1$ way.
- Total number of ways is then

$$C(11,1)\times C(10,2)\times C(8,4)\times C(4,4) = 11\times 45\times 70\times 1 = 34{,}650.$$

Combinations with repetitions

Task: select r items out of a set of size n.

- Repetitions allowed, order matters: n^r.
- Repetitions not allowed, order matters: $P(n,r)$.
- Repetitions not allowed, order doesn't matter: $C(n,r)$.
- What's missing? Repetitions allowed, order doesn't matter.

Example

A bakery is running a special: 13 cookies for the price of 12. They sell 6 kinds of cookies. In how many ways can a customer choose to order her 13 cookies? (Must assume bakery has at least 13 of each kind of cookie on hand.)

Combinations with repetitions (cont'd)

Suppose our bakery order is

Type	Chocolate chip	Oatmeal Raisin	Kale	Black & white	Sugar	Peanut butter
Number	3	2	0	5	2	1

Represent as

* * * | * * | | * * * * * | * * | *

- There are 13 asterisks to emplace.
- We have $13+6-1 = 18$ total slots, any of which can be an asterisk or a bar.
- So we're trying to place 13 asterisks into 18 slots, which can be done in $C(18,13) = 8{,}568$ ways.

Combinations with repetitions (cont'd)

Theorem

There are $C(n+r-1,r)$ ways to choose an r-element subset of an n-element set, if repetitions are allowed.

Proof.

- Represent each such choice as a list of $n-1$ bars (marking of n different cells) and and r asterisks; the jth cell containing one asterisk for each time the jth element of the set is chosen.
- There are $n+r-1$ total slots, each containing either an asterisk or a bar.
- We need to choose in which of these $n+r-1$ to place the r asterisks.
- So there are $C(n+r-1,r)$ ways to make this assignment. □

Combinations with repetitions (cont'd)

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$$

have, where $x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{N}$?

Solution: This is the same thing as our bakery problem! The answer is that there are $C(18, 13) = 8{,}568$ solutions to this problem.

Combinations with repetitions (cont'd)

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$$

have, where $x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{N}$, with $x_1 \geq 2$ and $x_3 \geq 1$?

Solution: This is like the bakery problem, except that we're only selecting $13 - (2 + 1) = 11$ items from among 6 items (with repetitions allowed). So there are $C(6 + 11 - 1, 11) = C(16, 11) = 4{,}368$ different possible solutions under the given constraints.

The selection problem: a summary

Use this table when determining how many ways we can select r elements from a set of size n:

	Order matters	Order doesn't matter
Repetitions allowed	n^r	$C(n + r - 1, r)$
Repetitions not allowed	$P(n, r)$	$C(n, r)$