CISC 1400—Discrete Structures
Tuesday 25 June 2017

PRACTICE FINAL EXAMINATION (SOLUTIONS)

1. Define functions $f, g\colon \mathbb{N} \to \mathbb{N}$ by

$$f(x) = 2x + 5 \qquad \text{and} \qquad g(x) = 3x + 2 \forall\, x \in \mathbb{N}\,.$$

(a) What is $g \circ f$?

Solution: For any $x \in \mathbb{N}$, we have

$$(g \circ f)(x) = g\big(f(x)\big) = g(2x + 5) = 3(2x + 5) + 2 = 6x + 15 + 2 = 6x + 17$$

(b) What is $f \circ g$?

Solution: For any $x \in \mathbb{N}$, we have

$$(f \circ g)(x) = f\big(g(x)\big) = f(3x + 2) = 2(3x + 2) + 5 = 6x + 4 + 5 = 6x + 9$$

2. Define a function $h\colon \mathbb{N} \to \mathbb{N}$ by

$$h(n) = 2n^2 + 3 \qquad \forall\, n \in \mathbb{N}.$$

(a) Is h injective? Why or why not?

Solution: Suppose that $h(m) = h(n)$ for some $m, n \in \mathbb{N}$. We then have

$$2m^2 + 3 = 2n^2 + 3 \Longrightarrow 2m^2 = 2n^2 \Longrightarrow m^2 = n^2.$$

Now $m^2 = n^2$ iff $m = n$ or $m = -n$. But since $m, n \in \mathbb{N}$, we see that we can eliminate the choice $m = -n$. Thus $m = n$. Since we have shown that $h(m) = h(n) \Longrightarrow m = n$, the function h is injective.

(b) Is h surjective? Why or why not?

Solution: Note that $h(0) = 3$ and $h(1) = 5$. Since $h(n)$ increases with n, there is no $n \in \mathbb{N}$ such that $h(n) = 4$. Thus 4 is not in the range of h, and so h is not surjective.

3. Let $f\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 9}$ be defined as

$$f(x) = 2x^2 + 9 \qquad \forall\, x \in \mathbb{R}^{\geq 0}.$$

This function is invertible. Find its inverse.

Solution: If $y \geq 9$, then $y = f(x)$ for some $x \in \mathbb{R}^{\geq 0}$ iff $2x^2 + 9 = y$, which in turn holds iff $x^2 = \frac{1}{2}(y - 9)$, the latter holding iff $x = \sqrt{\frac{1}{2}(y - 9)}$. (Here, we use the fact that $x \geq 0$ to not have a $\pm$ in front of the square root sign, along with the fact that $y \geq 9$ allows us to take this square root.) So the inverse function $f^{-1}\colon \mathbb{R}^{\geq 9} \to \mathbb{R}^{\geq 0}$ is given by

$$f^{-1}(y) = \sqrt{\tfrac{1}{2}(y - 9)} \qquad \forall\, y \in \mathbb{R}^{\geq 9}\,.$$

4. Suppose that there are ten applicants (seven men and three women) for a job. The women will be interviewed before the men; however both the women and the men will be interviewed in random order. In how many ways can the applicant be interviewed?

Solution: The women can be interviewed in

$$P(3,3) = 3! = 6$$

different ways, and the men can be interviewed in

$$P(7,7) = 7! = 5040$$

different ways. Since the women are being interviewed before the men, the full set of applicants can be interviewed in

$$P(3,3) \cdot P(7,7) = 6 \cdot 5040 = 30240$$

different ways.

5. Compute each of the following.

(a) $C(5,0)$. **Solution:** 1

(b) $C(5,1)$. **Solution:** 5

(c) $C(5,2)$. **Solution:** 10

(d) $C(1000,1)$. **Solution:** 1000

(e) $C(1000,1000)$. **Solution:** 1

6. One of my favorite pizzerias sells pizzas with the following toppings: pineapple, felafel, mushrooms, green peppers, and onions.

(a) Taking account of all possible situations (i.e., you might decide to order any number of these toppings, or none at all), how many different kinds of pizzas can you order?

Solution: Each of the five toppings is independent of the other. Since there are two choices for each topping (i.e., whether or not to get same), there are $2^5 = 32$ different kinds of pizza.

(b) The pizzeria is running a post-exam special, where a customer can get up to two toppings for free. How many different kinds of such pizzas (i.e., with *at most* two toppings) are there?

Solution: For any $k \in \{0,1,2\}$, there are $C(5,k)$ different pizzas having exactly k of the 4 toppings. Since

$$\begin{aligned} C(5,0) &= 1 \\ C(5,1) &= 5 \\ C(5,2) &= \frac{5 \cdot 4}{2 \cdot 1} = 10, \end{aligned}$$

we see that there are

$$C(5,0) + C(5,1) + C(5,2) = 1 + 5 + 10 = 16$$

pizzas having at most two toppings.

7. In (five-card draw) poker, each player is dealt five cards out of a 52-card deck.

(a) How many different poker hands are there?

Solution: There are

$$C(52,5) = \frac{52!}{5!\cdot 47!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2}$$
$$= 52\cdot 51\cdot 5\cdot 49\cdot 4 = 2598960$$

different hands.

(b) In how many ways can one get a straight flush, i.e., a hand in which all the cards are of the same suit and in sequential order, i.e., it's both a straight and a flush.

Solution: The cards in a straight flush are uniquely determined by the bottom card (i.e., the card having the lowest denomination [since it's a straight] and the suit (since it's a flush). The bottom card can be `2`, `3`, ..., `10`, which means that there are nine possible bottom cards within any given suit. Since there are four suits, it follows that one get a straight flush in $9 \times 4 = 36$ different ways.

(c) What is the probability of drawing a straight flush in five-card draw poker?

Solution: The probability is

$$\frac{36}{2598960} = \frac{3}{649490} = 0.000013857.$$

8. A red hat contains ten numbers (`1`, `2`, ..., `10`) and a blue hat contains ten letters (`A`, `B`, ..., `J`). You draw one item from each hat. You will win a prize if the number is less than or equal to 4 or the letter is an `A`, `B`, or `C` (or both occur). What is the probability that you win a prize?

Solution: Solution: Let E_1 and E_2 denote the events "the number is less than or equal to 4" and "the letter is an `A`, `B`, or `C`", respectively. Then

$$\text{Prob}(E_1) = \tfrac{4}{10} = \tfrac{2}{5} \qquad \text{and} \qquad \text{Prob}(E_2) = \tfrac{3}{10}.$$

Since E_1 and E_2 are independent, we have

$$\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1)\cdot\text{Prob}(E_2) = \tfrac{2}{5}\cdot\tfrac{3}{10} = \tfrac{6}{50} = \tfrac{3}{25}.$$

So

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2) = \tfrac{2}{5} + \tfrac{3}{10} - \tfrac{3}{25} = \tfrac{29}{50} = 0.58.$$

9. What is the probability of flipping a coin ten times and getting a total of five heads and five tails?

Solution: The probability of flipping a coin ten times and getting a total of five heads and five tails can be computed using the formula associated with the Bernoulli distribution, since each flip of the coin is a Bernoulli trial. The general formula is:

$$\text{Prob}(K = k) = C(n,k)p^k(1-p)^{n-k}$$

Although the situation is symmetric, we should still decide on whether we focus on the heads or tails. Let's focus on heads. Since there are ten flips and five heads, we have $n = 5$ and $k = 5$, and so

$$\text{Prob}(k = 5) = C(10,5)\cdot(\tfrac{1}{2})^5\cdot(\tfrac{1}{2}^5) = 252\cdot(\tfrac{1}{2})^{10} = \tfrac{252}{1024} \doteq 0.2461.$$

10. Trace the bubble sort algorithm as it sorts the set $\{30, 10, 20, 50, 40, 70, 80, 60\}$.
Solution: The vertical bar separates the not-yet-sorted and the sorted regions.

30 10 20 50 40 70 80 60 |
10 20 30 40 50 70 60 | 80
10 20 30 40 50 | 60 70 80
10 20 30 40 | 50 60 70 80
10 20 30 | 40 50 60 70 80
10 20 | 30 40 50 60 70 80
10 | 20 30 40 50 60 70 80

Note that the smarter version of bubblesort would stop early.

11. Lack of time prevents me from giving you a fully-defined graph theory example. But be able to do the following:

(a) We know several ways of describing a graph:

- A list V of vertices and a list E of edges, the latter being either of the form $\{v, w\}$ (for undirected graphs) or (v, w) (for digraphs), where $v, w \in E$.
- A drawing.
- An incidence matrix.

Given any such representation, you should be able to produce the other two. This is for both unweighted and weighted graphs.

(b) Determine whether there are any Euler trails or circuits.

(c) Determine whether there are any Hamiltonian circuits (for small graphs only).

(d) Find a minimal spanning tree.

(e) Find the reachability matrix.

So make up an example of such a graph, and see if you can do these operations. If you're stuck thinking of something sufficiently simple, pick one of the graphs in the book.