CISC 1100: Structures of Computer Science

Chapter 2
Sequences

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Summer, 2016

Outline

- Finding patterns
- Notation
  - Closed form
  - Recursive form
  - Converting between them
- Summations

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15, 21
- 1, 2, 6, 24, 120, 720
- 1, 1, 2, 3, 5, 8, 13, 21

Discovering the pattern

- Each term might be related to previous terms
- Each term might depend on its position number (1st, 2nd, 3rd, . . .)
- "Well-known" sequences (even numbers, odd numbers)
- Some (or all) of the above

2, 4, 6, 8, 10, ...

Can we relate a term to previous terms?

- Second term is 2 more than first term.
- Third term is 2 more than second term.

  ⋮

- Any given term is 2 more than previous term.

2, 4, 6, 8, 10, ...

Can we describe each term by its position in the sequence?

- Term at position 1 is 2.
- Term at position 2 is 4.
- Term at position 3 is 6.

  ⋮

- Term at position n is $2n$.

Mathematical notation

- Write term in a sequence as a lower case letter, followed by a *subscript* denoting position number of the term (e.g., a_1, b_7, z_k).
- For the sequence 2, 4, 6, 8, 10, ...:
  - $a_1 = 2$.
  - $a_2 = 4$.
  - a_n is nth term in the sequence.
- What is a_3? 6
- What is a_5? 10
- What is a_6? 12
- What is a_n if $n = 5$? 10
- What is a_{n+1} if $n = 5$? 12

Recursive formula

- *Recursive formula* for a sequence: each term is described in relation to previous term(s). For example:

  $$a_n = 2a_{n-1}$$

  So

  $$\begin{aligned} a_3 &= 2a_2 = 2 \cdot (2\,a_1) = 4a_1 = 4 \cdot (2a_0) = 8a_0 \\ &= 8 \cdot (2a_{-1}) = 16a_{-1} = \dots \end{aligned}$$

- Problem: Need a starting point (initial condition) such as

  $$a_1 = 1$$

- So let's try

  $$\begin{aligned} a_n &= 2a_{n-1} \qquad \text{for } n \geq 2 \\ a_1 &= 1 \end{aligned}$$

- Example:

  $$a_3 = 2a_2 = 2 \cdot (2\,a_1) = 4a_1 = 4 \cdot 1 = 4$$

Fibonacci sequence

- $1, 1, 2, 3, 5, 8, 13, \ldots$
- Recursive formula:

$$a_n = a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3$$
$$a_2 = 1$$
$$a_1 = 1$$

- What's a_{10}? Top-down solution:

$$a_{10} = a_9 + a_8 = (a_8 + a_7) + (a_7 + a_6) = a_8 + 2a_7 + a_6 \ldots$$

  Too hard!
- Better way? Work bottom-up via a grid.

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	3	5	8	13	21	34	55

Recursion

- Recursive formula corresponds to "recursive function" in a programming language.
- Fibonacci formula

$$a_n = a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3$$
$$a_2 = 1$$
$$a_1 = 1$$

- Recursive function

```
def fib(n):
    if n==1 or n==2:
        return 1
    else:
        return fib(n-1) + fib(n-2)
```


Exercise: Find recursive formula

- 2, 4, 6, 8, 10, ...

$$a_n = a_{n-1} + 2 \qquad \text{for } n \geq 2$$
$$a_1 = 2$$

- 1, 3, 6, 10, 15, ...

$$a_n = a_{n-1} + n \qquad \text{for } n \geq 2$$
$$a_1 = 1$$

- 2, 2, 4, 6, 10, 16, ...

$$a_n = a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3$$
$$a_2 = 2$$
$$a_1 = 2$$

Finding a closed formula

- Write each term in relation to its position
- Example: 2, 4, 6, 8, 10, ...
  - $a_1 = 2 = 2 \cdot 1$
  - $a_2 = 4 = 2 \cdot 2$
  - $a_3 = 6 = 2 \cdot 3$
  - More generally, $a_n = 2n$.

Find the closed formulas

- 1, 3, 5, 7, 9, $\ldots a_n = 2n - 1$
- 3, 6, 9, 12, 15, $\ldots b_n = 3n$
- 8, 13, 18, 23, 28, $\ldots c_n = 5n + 3$
- 3, 9, 27, 81, 243, $\ldots d_n = 3^n$

Recursive formulas vs. closed formulas

- Recursive formula
  - It's often easier to find a recursive formula for a given sequence.
  - It's often harder to evaluate a given term.
- Closed formula
  - It's often harder to find a closed formula for a given sequence.
  - It's often easier to evaluate a given term.

Closed formula $\Rightarrow$ recursive formula

- Write out a few terms.
- See if you can figure out how a given term relates to previous terms.
- Example: $r_n = 3n + 4$.

n	1	2	3	4	5	...
r_n	7	10	13	16	19	...

We find

$$\begin{aligned} r_n &= r_{n-1} + 3 \qquad \text{for } n \geq 2 \\ r_1 &= 7 \end{aligned}$$

Closed formula $\Rightarrow$ recursive formula

Can also use algebraic manipulation. Let's try

$$r_n = 3n + 4$$

again.

- Initial condition is easiest—substitute $n = 1$ into closed form:

$$r_1 = 3 \cdot 1 + 4 = 7$$

- Recursive formula: Try to describe r_n in terms of r_{n-1}:

$$\begin{aligned} r_n &= 3n + 4 \\ r_{n-1} &= 3(n-1) + 4 = 3n - 3 + 4 = 3n + 1 \end{aligned}$$

So

$$r_n - r_{n-1} = (3n + 4) - (3n + 1) = 3,$$

i.e.,

$$r_n = r_{n-1} + 3$$

Another example

$$s_n = 2^n - 2$$

- Initial condition: $s_1 = 2^1 - 2 = 0$.
- Recursive formula: We have

  $$s_n = 2^n - 2$$

  and

  $$s_{n-1} = 2^{n-1} - 2$$

  So

  $$\begin{aligned} s_n &= 2^n - 2 = 2 \cdot 2^{n-1} - 2 = 2 \cdot 2^{n-1} - 4 + 2 \\ &= 2 \cdot (2^{n-1} - 2) + 2 \\ &= 2s_{n-1} + 2 \end{aligned}$$

Exercise

Find the recursive formulas for the following sequences:

- $a_n = 2n + 7$
  - $a_1 = 9$
  - $a_n = a_{n-1} + 2$ for $n \geq 2$.
- $b_n = 2^n - 1$
  - $b_1 = 1$
  - $b_n = 2b_{n-1} + 1$ for $n \geq 2$.

Summations

Summing the terms in a sequence: important enough to have its own notation ("sigma notation"):

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Parts of speech?

- Large Σ: "summation"
- $i = 1$ at bottom: We want to start summation at term #1 of the sequence.
- n at the top: We want to stop summation at the nth term of the sequence
- Portion to the right of the $\Sigma_{i=1}^{n}$: Closed form of sequence we want to sum.

Examples of Σ-notation:

$$\begin{aligned} \sum_{i=1}^{5}(3i+7) &= (3 \cdot 1 + 7) + (3 \cdot 2 + 7) + (3 \cdot 3 + 7) + (3 \cdot 4 + 7) \\ &\quad + (3 \cdot 5 + 7) \\ &= 10 + 13 + 16 + 19 + 22 = 80 \end{aligned}$$

$$\begin{aligned} \sum_{j=2}^{6}(j^2 - 2) &= (2^2 - 2) + (3^2 - 2) + (4^2 - 2) + (5^2 - 2) + (6^2 - 2) \\ &= 2 + 7 + 14 + 23 + 34 = 80 \end{aligned}$$

Note: Parentheses are important!

$$\sum_{i=1}^{5} 3i + 7 = (3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 3 \cdot 5) + 7 = 52$$

Converting a sum into Σ-notation

$$\begin{aligned} 3+7+11+15+19 &= \sum_{i=1}^{5}(4i-1) \\ &= \sum_{j=1}^{5}(4j-1) \\ 0+3+8+15+24 &= \sum_{k=1}^{5}(k^2-1) \end{aligned}$$