CISC 1100: Structures of Computer Science

Chapter 6
Counting

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Why talk about counting in a college-level course?

- Counting isn't as easy as it looks.
  - Simple sets: trivial to count.
  - Complicated sets: hard to count.
    - Facebook FOAF.
    - Number of ways to fill a committee.
    - Number of ways to fill a slate of officers.
    - Number of outcomes in a game (chess, poker, ...).
  - Methodically enumerating a set.
- Connection between counting and probability theory.

Outline

- Counting and how to count
- Elementary rules for counting
  - The addition rule
  - The multiplication rule
  - Using the elementary rules for counting together
- Permutations and combinations
- Additional examples

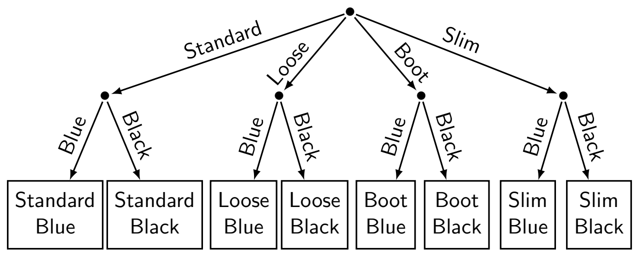
Counting and how to count

- Some things are easy to count (e.g., number of students in this class).
- Some things are harder to count.
- **Example:** You are asked to select a pair of men's jeans.
  - Four styles are available (standard fit, loose fit, boot fit, and slim fit).
  - Each style comes in two colors (blue or black).
- You *could* list all possibilities for this problem.

	Jeans Style			
Color	Standard	Loose	Boot	Slim
Blue	Standard-Blue	Loose-Blue	Boot-Blue	Slim-Blue
Black	Standard-Black	Loose-Black	Boot-Black	Slim-Black

Counting and how to count (cont'd)

- This doesn't generalize.
  - What if more than two "features"?
- One idea: Use a *tree structure* to help you enumerate the choices.



Counting and how to count (cont'd)

Example: We toss a penny, a nickel, and a dime into the air. How many different configurations?

- How to encode? As a triple:

  (penny's state, nickel's state, dime's state)

- Configurations?

$$\begin{aligned} C = \{&(H,H,H), (H,H,T), (H,T,H), (H,T,T), \\ &(T,H,H), (T,H,T), (T,T,H), (T,T,T)\}\,. \end{aligned}$$

- How many configurations? 8.
- How to count configurations without listing?

Elementary rules of counting

- Two basic rules:
  - Addition rule
  - Multiplication rule
- Using these rules together

Elementary rules of counting: the addition rule

- **Example:** You need to purchase one shirt of any kind. The store has five short sleeve shirts and eight long sleeve shirts. How many possible ways are there to choose a shirt?
- **Solution:** $8 + 5 = 13$.
- **Addition rule:**
  - If we have two choices C_1 and C_2, with C_1 having a set O_1 of possible outcomes and C_2 having a set O_2 of possible outcomes, with $|O_1| = n_1$ and $|O_2| = n_2$, then the total number of outcomes for C_1 *or* C_2 occurring is $n_1 + n_2$.
  - If we have k choices $C_1, \dots, C_k$ having $n_1, \dots, n_k$ possible outcomes, then the total number of ways of C_1 occurring *or* C_2 occurring *or* ... *or* C_k occurring is $n_1 + n_2 + \cdots + n_k$.
- Fairly straightforward.

Elementary rules of counting: the multiplication rule

- In our jeans example,
$$\# \text{ of jeans configurations} = (\# \text{ number of styles}) \times (\# \text{ of colors})$$
- **Multiplication rule:**
  - If we have two choices C_1 and C_2, with C_1 having a set O_1 of possible outcomes and C_2 having a set O_2 of possible outcomes, with $|O_1| = n_1$ and $|O_2| = n_2$, then the total number of possible outcomes for C_1 *and* C_2 occurring is $n_1 \times n_2$.
  - More generally, if we have k choices $C_1, \ldots, C_k$ having $n_1, \ldots, n_k$ possible outcomes, then the total number of ways of C_1 occurring *and* C_2 occurring *and* ... *and* C_k occurring is $n_1 \times n_2 \times \cdots \times n_k$.
- Roughly speaking:
  - addition rule: "or" rule
  - multiplication rule: "and" rule

Elementary rules of counting: the multiplication rule (cont'd)

Example: Solve jeans problem via multiplication rule ...
- four styles (standard, loose, slim, and boot fits) and
- two colors (black, blue)

Solution: Our choices?
$$C_1 = \text{"choose the jeans style"},$$
$$C_2 = \text{"choose the jeans color"}.$$
Our outcomes?
$$O_1 = \{\text{standard fit, loose fit, boot fit, slim fit}\},$$
$$O_2 = \{\text{black, blue}\}.$$
Now determine the cardinalities of the sets:
$$n_1 = |O_1| = 4 \qquad n_2 = |C_2| = 2.$$
Now we apply the multiplication rule
$$\text{Total number of outcomes} = n_1 \times n_2 = 4 \times 2 = 8.$$

Elementary rules of counting: the multiplication rule (cont'd)

- Why does the multiplication rule work?
- The set of possible outcomes is for O_1 *and* O_2 occurring is $O_1 \times O_2$.
- We know that $|O_1 \times O_2| = |O_1| \cdot |O_2|$.
- This is the multiplication rule!

Elementary rules of counting: the multiplication rule (cont'd)

Example: Suppose that you flip a coin twice and record the outcome (head or tail) for each flip. How many possible outcomes are there?
Solution: There are two choices, C_1 and C_2, corresponding to the two coin flips. C_1 *and* C_2 must occur, so the multiplication rule applies. Each choice has two possible outcomes, thus $n_1 = 2$ and $n_2 = 2$. Thus by the multiplication principle of counting, there are $2 \times 2 = 4$ possible outcomes.

Elementary rules of counting: the multiplication rule (cont'd)

Example: You are asked to flip a coin five times and to record the outcome (head or tail) for each flip. How many possible outcomes are there?

Solution:

- This example differs from the previous one only in that there are five choices instead of two.
- For each choice there are two possible outcomes.
- The total number of outcomes is

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. If a number may be selected more than once, then how many ways can you fill out the lottery card?

Solution:

- There are five choices, corresponding to the five numbers that you must choose.
- Each of the five choices must occur, so the multiplication rule applies.
- Each choice has twenty possible outcomes (i.e., you pick a number between 1 and 20).
- There are

$$20 \times 20 \times 20 \times 20 \times 20 = 20^5 = 3{,}200{,}000$$

possible ways to fill out the lottery card.

Elementary rules of counting: the multiplication rule (cont'd)

Example: You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. You must choose the numbers in the order that they appear in the winning selection. The numbers are chosen by the lottery commission from a bin and once a number is chosen it is discarded and cannot be chosen again. In how many ways can you fill out the lottery card?

Elementary rules of counting: the multiplication rule (cont'd)

Solution:

- Close to the previous one, but a number cannot be chosen more than once.
- Hence, the number of possible outcomes for each choice is progressively reduced by one.
- Number the five choices $C_1 \ldots C_5$ such that C_1 corresponds to the first number selected and C_5 to the last number selected.
- The number of outcomes for C_1 is 20, for C_2 is 19, for C_3 is 18, for C_4 is 17 and for C_5 is 16.
- Thus the number of possible outcomes is

$$20 \times 19 \times 18 \times 17 \times 16 = 1{,}860{,}480.$$

Elementary rules of counting: the multiplication rule (cont'd)

- Don't be misled by the word "and"!
- **Example:** How many ways are there to choose one class among 5 day classes and 2 evening classes?
- **Solution:** $5 + 2 = 7$ ways.

Elementary rules of counting: combining the rules together

- **Example:** How many odd three-digit numbers are there (allowing leading zeros, such as 007)?
- **First solution:**
  - We have three choices, one per digit. Let C_1, C_2, C_3 denote the choices for the first, second, third digits.
  - $O_1 = O_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, while $O_3 = \{1, 3, 5, 7, 9\}$.
  - So $|O_1| = 10$, $|O_2| = 10$, $|O_3| = 5$.
  - Hence there are $10 \times 10 \times 5 = 500$ outcomes.

Elementary rules of counting: combining the rules together

- **Example:** How many odd three-digit numbers are there (allowing leading zeros, such as 007)?
- **Second solution:**
  - Number of outcomes = number of outcomes where the three-digit number ends in a 1 *or* 3 *or* 5 *or* 7 *or* 9.
  - Each of these five cases has $10 \times 10 = 100$ outcomes.
  - So there are $5 \times 100 = 500$ outcomes overall.

Facts about playing cards

- A deck of cards contains 52 cards.
- Each card belongs to one of four *suits*

  $\clubsuit$ (Clubs), $\diamondsuit$ (Diamonds), $\heartsuit$ (Hearts), $\spadesuit$ (Spades)

  and one of thirteen denominations

  2, 3, 4, 5, 6, 7, 8, 9, 10, J(ack), Q(ueen), K(ing), A(ce).
- The clubs and spades are black and the diamonds and hearts are red.
- Unless otherwise specified, assume that for any example you begin with a complete deck and that as cards are dealt they are not immediately replaced back into the deck.
- We abbreviate a card using the denomination and then suit, such that $2\heartsuit$ (or 2H) represents the 2 of Hearts.

Poker hands

- In standard poker you receive 5 cards.
- The suits are equally important.
- The face values are ordered
  $2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < J < Q < K < A$
- While you can later discard cards and then replace them, for most of our examples we will only consider the initial configuration.
- *Pair* (*two of a kind*): two cards that are the same denomination, such as a pair of 4's.
- *Three of a kind* and *four of a kind* are defined similarly.
- *Full house*: three of one kind and a pair of another kind.
- *Straight*: the cards are in sequential order, with no gaps.
- *Flush*: all five cards are of the same suit.
- *Straight flush*: all five cards are of the same suit and in sequential order (i.e., a straight *and* a flush).

Poker hands (cont'd)

Ordering of the hands (highest to lowest):

- straight flush (with a "royal flush" [ace high] the highest possible hand of all)
- four of a kind
- full house
- flush
- straight
- three of a kind
- two pairs
- one pair
- high card

A poker example

In how many ways can you draw a flush in poker, assuming that the order of the five cards drawn matters? (We will learn how to relax this assumption in the next section.)

- There are four basic ways to get a flush: all clubs or all diamonds or all hearts or all spades.
- Each is an outcome satisfying the condition of drawing a flush; we want to determine the total number of outcomes of these four *non-overlapping* outcomes.
- How many ways can we get an all-clubs flush? By multiplication rule to select 5 cards without replacement,

$$\#\text{ ways to draw five clubs} = 13 \times 12 \times 11 \times 10 \times 9 = 154{,}440.$$

- Therefore, by the addition rule, there are $4 \times 154{,}440 = 617{,}760$ ways to get a flush.

Permutations and Combinations

- Sometimes order matters, sometimes it doesn't.
- **Example:** How many ways to get a royal flush in spades?

$$A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit$$

  - If order matters, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.
  - If order does not matter, there is only 1 way.
- Order matters: *permutation*
- Order doesn't matter: *combination*

Permutations

- *Permutation:* order matters, cannot reuse objects.
- Phone numbers 123–456–7890 and 789–012–3456 are different. These are two permutations of the set of digits.
- **Example:** How many ways to seat 5 children in 5 chairs?
  - Both criteria for permutations are satisfied.
  - Counting permutations of 5 children.
  - By multiplication rule, there are

    $$5 \times 4 \times 3 \times 2 \times 1 = 120$$

    different seating arrangements.
- **Example:** How many ways to seat 10 children in 5 chairs?
  - Both criteria for permutations are satisfied.
  - Counting permutations of 10 children, *chosen 5 at a time.*
  - By multiplication rule, there are

    $$10 \times 9 \times 8 \times 7 \times 6 = 30{,}240$$

    different seating arrangements.

Permutations (cont'd)

- **Notation:** $P(n, r)$ is the number of permutations of n objects, chosen r at a time.
- Formula for $P(n, r)$?

  $$P(n, r) = n(n-1)(n-2)\dots(n-r+1)$$

- Excursus on factorials
  - $n!$ is the product of the natural numbers $1, 2, \dots, n$.
  - Semi-special case: $0! = 1$.
  - Table of factorials:

n	0	1	2	3	4	5	6	7	...
$n!$	1	1	2	6	24	120	720	5,040	...

n	...	8	9	10	...
$n!$	...	40,320	362,880	3,628,800	...

- "Simplified" formula for $P(n, r)$?

  $$P(n, r) = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Permutations (cont'd)

- **Example (cont'd):** We have

  $$P(10, 5) = 10 \times 9 \times 8 \times 7 \times 6 = 30{,}240.$$

- We also have

  $$P(10, 5) = \frac{10!}{(10-5)!} = \frac{10!}{5!}.$$

- Save some work: cancel common factors

  $$\begin{aligned} P(10, 5) &= \frac{10!}{(10-5)!} = \frac{10!}{5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\ &= 10 \times 9 \times 8 \times 7 \times 6 = 30{,}240. \end{aligned}$$

- All our answers agree.

Permutations (cont'd)

- Sanity check:
  - $P(n, r)$ counts something.
  - Thus $P(n, r)$ *must* be a non-negative integer.
  - The formula

    $$P(n, r) = \frac{n!}{(n-r)!}$$

    appears to involve division.
  - You will *always* be able to use the cancellation trick to get rid of divisions.
  - Alternatively, use the formula

    $$P(n, r) = n(n-1)(n-2)\dots(n-r+1).$$

    (There are r factors.)
- If the answer you get to a permutation problem is anything other than a non-negative integer, *go back and check your work*!

Permutations (cont'd)

- **Example:** In how many ways can we choose a 3-person slate of officers (president, vice-president, secretary) out of the 10 members in this class?
- **Solution:** We need to choose 3 distinct people out of 10, with order mattering.
- So
$$P(10,3) = 10 \times 9 \times 8 = 720.$$

Permutations (cont'd)

- **Example:** In major league baseball, each team has a 25-player roster. How many possible batting orders are there for such a roster?
- **Solution:** Check that this is a permutation.
- Total number of batting orders is
$$P(25,9) = \frac{25!}{16!} = 25 \times 24 \times \cdots \times 17 = 741{,}354{,}768{,}000.$$

Combinations

- For some problems, order matters. (Baseball lineup problem.)
- For some problems, order does *not* matter.
- **Example:** We need to choose a 12-person jury from a pool of 1000 people. The order does not matter here. We want the number of *combinations* of 1000 persons, chosen 12 at a time.
- **Notation:** $C(n,r)$ denotes the number of *combinations* of n objects, chosen r at a time. Here the order *does not* matter, and we are not allowed to reuse objects. We often read this as "n choose r".
- Formula for combinations:
$$C(n,r) = \frac{n!}{(n-r)!r!}$$

  Why?
$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!}$$

Combinations (cont'd)

- **Example:** In how many ways can we choose a 3-person committee out of a 10-member class?
- **Solution:** We need to choose 3 distinct people out of 10, with order not mattering.
- So
$$\begin{aligned} C(10,3) &= \frac{10!}{3! \cdot 7!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)} \\ &= \frac{10 \times \cancelto{3}{9} \times \cancelto{4}{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{(\cancel{3} \times \cancel{2} \times \cancel{1})(\cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1})} \\ &= 10 \times 3 \times 4 = 120. \end{aligned}$$

Combinations (cont'd)

- Sanity check:
  - $C(n,r)$ counts something.
  - Thus $C(n,r)$ *must* be a non-negative integer.
  - The formula
    $$C(n,r) = \frac{n!}{(n-r)!r!}$$
    involves division.
  - You will *always* be able to use the cancellation trick to get rid of divisions.
- If the answer you get to a combination problem is anything other than a non-negative integer, *go back and check your work*!

Additional Examples

Example: A typical telephone number has 10 digits (e.g., 555–817–4495), where the first three are known as the area code and the next three as the exchange.

1. Assuming *no* restrictions, how many possible (three-digit) area codes are there?
   Solution: $10 \times 10 \times 10 = 1{,}000$ three-digit area codes.
2. Assuming that the middle digit of the area code must be a 0 or a 1 (which was required until recently), how many possible (3 digits) area codes are there?
   Solution: $10 \times 2 \times 10 = 200$ area codes.
3. Assuming no restrictions whatsoever, how many possible values are there for the full 10-digit phone number?
   Solution: $10^{10} = 10{,}000{,}000{,}000$ phone numbers
4. If the only restriction is that no digit may be used more than once, how many possible 10-digit phone numbers are there?
   Solution: $10 \times 9 \times \cdots \times 1 = 10! = 3{,}628{,}800$ phone numbers.

Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt "two pairs"?

Solution:

- There are $C(13,2)$ ways to identify the two denominations.
- For each denomination, there are $C(4,2)$ ways to choose two of the four cards. Do this twice.
- Pick the last card? 11 ways for each of 4 suits.
- Final answer:

$$C(13,2) \times C(4,2) \times C(4,2) \times 11 \times 4 = 123{,}552 \text{ ways.}$$

Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt "three of a kind"?

Solution:

- We can choose the denomination with 3 of a kind in 13 ways.
- There are $C(4,3)$ ways to choose the three cards of said denomination.
- The two remaining cards must come from the other 12 denominations. They can't be the same, since this would yield a full house. Since there are 4 suits, there are $C(12,2) \times 4 \times 4$ ways of choosing these two cards.
- Final answer:

$$13 \times C(4,3) \times C(12,2) \times 4 \times 4 = 54{,}912 \text{ ways.}$$

Additional Examples (cont'd)

A poker player is dealt a hand of 5 cards from a freshly mixed deck. In how many ways can one be dealt a "full house"?

Solution:

- A full house requires 3 of a kind and also 2 of a different kind.
- We can choose the denomination with 3 of a kind in 13 ways and then we can choose the 3 specific cards in $C(4, 3)$ ways.
- Then we can choose the denomination with the 2 of a kind in 12 ways and choose the 2 specific cards in $C(4, 2)$ ways.
- Final answer:

$$13 \times C(4, 3) \times 12 \times C(4, 2) = 3{,}744 \text{ ways.}$$

Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word `MISSISSIPPI`?

Solution #1:

- Multiplication rule: 11! ways.
- Since there are
  - 4 instances of `S` and `I`
  - 2 instances of `P`

  not all 11! ways are distinguishable.
- Since there are 4 instances of `S`, their appearance can be permuted in 4! different ways. So we need to divide the current answer by 4!, getting $11!/4!$.
- Since there are 4 instances of `I`, their appearance can be permuted in 4! different ways. So we need to divide the current answer by 4!, getting $11!/(4!4!)$.

Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word `MISSISSIPPI`?

Solution #1 (contd):

- Answer so far (accounting for multiple `S` and `I`): $11!/(4!4!)$.
- Since there are 2 instances of `P`, their appearance can be permuted in 2! different ways. So we need to divide the current answer by 2!, getting $11!/(4!4!2!)$.
- Final answer:

$$\frac{11!}{4!4!2!} = 11 \times 10 \times 9 \times 7 \times 5 = 34{,}650 \text{ ways.}$$

Additional Examples (cont'd)

How many distinguishable ways are there to arrange the letters in the word `MISSISSIPPI`?

Solution #2: Use a "fill-in-the-blank" approach, starting with 11 blanks

___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

- Can assign the one `M` in $C(11, 1) = 11!/(10! \times 1!) = 11$ ways.
- Can assign the two `P`'s in $C(10, 2)$ ways.
- Can assign the four `S`'s in $C(8, 4)$ ways.
- Can assign the four `I`'s in $C(4, 4) = 1$ way.
- Total number of ways is then

$$C(11, 1) \times C(10, 2) \times C(8, 4) \times C(4, 4) = 11 \times 45 \times 70 \times 1 = 34{,}650.$$