CISC 1100: Structures of Computer Science

Chapter 7
Probability

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Why study probability?

Want to know the likelihood of some event:

- Getting a "head" when flipping a coin (should be $\frac{1}{2}$)
- Getting at least two "heads" when flipping a coin four times
- Getting 3 when rolling a six-sided die (should be $\frac{1}{6}$)
- Getting a 7 when rolling two dice
- Winning at poker
  - Getting a particular hand on the initial deal
  - Completing a hand in draw poker
- Winning PickSix, PowerBall, MegaMillions
- Rain (should I bring an umbrella)?
- The economy improving
- A nuclear power plant failing

Historical note: A gambler's dispute in 1654 led Blaise Pascal and Pierre de Fermat to create a mathematical theory of probability.

Outline

- Terminology and background
- Complement
- Elementary rules for probability
- General rules for probability
- Bernoulli trials and probability distributions
- Expected value

Terminology and background

- An *experiment* will have *outcomes*
  - Tossing a coin once: H, T
  - Tossing a coin four times: HHHH, HHHT, HHTH, ..., TTTT
  - Drawing a card from a deck: $2\clubsuit, 2\diamondsuit, \ldots, A\heartsuit, A\spadesuit$
  - PickSix possibilities
- *Event*: set E of desired outcomes
- *Sample space*: (finite) set S of all possible outcomes
- The *probability* $\text{Prob}(E)$ of an event $E \subseteq S$ is given as

$$\text{Prob}(E) = \frac{|E|}{|S|}$$

Terminology and background (cont'd)

Example: What is the probability of getting 7 when rolling two dice?

Solution:

- Our sample space S is

$$\begin{aligned} S = \{&(1,1),(1,2),\ldots,(1,6),\\ &(2,1),(2,2),\ldots,(2,6),\\ &(3,1),(3,2),\ldots,(3,6),\\ &\vdots\\ &(6,1),(6,2),\ldots,(6,6)\}. \end{aligned}$$

- The event E is given by

$$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

- So

$$\text{Prob}(E) = \frac{|E|}{|S|} = \tfrac{6}{36} = \tfrac{1}{6} = 0.1667\,.$$

Terminology and background (cont'd)

Example: What is the probability of getting 7 when rolling two dice (cont'd)?

Discussion:

- Since E and S are small, could solve by enumeration.
- If either E or S is big, this is impractical. (Poker problems, lottery problems, ...).
- Can often use counting principles from previous chapter to determine $|S|$ and/or $|E|$.
- In our case, there are 6 outcomes for the roll of each of the two dice.
- So multiplication principle tells us that there are $6 \times 6 = 36$ outcomes for the roll of both dice, i.e., $|S| = 36$.

Terminology and background (cont'd)

- Since $0 \le |E| \le |S|$ and $\text{Prob}(E) = |E|/|S|$, we have

$$0 \le \text{Prob}(E) \le 1$$

.
  - $\text{Prob}(E) = 0$: event will never happen
  - $\text{Prob}(E) = 1$: event will certainly happen
- **Example:** For rolling two dice

$$\text{Prob(roll value is positive)} = 1$$
$$\text{Prob(roll value is negative)} = 0$$

- If you ever calculate a probability as being negative or being greater than 1, you've made a mistake.

Terminology and background (cont'd)

- Our definition

$$\text{Prob}(E) = \frac{|E|}{|S|}$$

relies on two assumptions:
  - $|S|$ is finite.
  - All observations are equally likely.
- If these don't hold, the formula is incorrect.
  - Example: Throwing a loaded die.
  - Example: Choosing a ball out of a bag, where some balls are larger than others.

Terminology and background (cont'd)

Determine the probability that you flip a coin three times and that you get all heads.

Solution: We can directly enumerate S and E:

$$S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$$
$$E = \{(H,H,H)\}$$
$$\text{Prob}(E) = \frac{|E|}{|S|} = \frac{1}{8} = 0.125\,.$$

Terminology and background (cont'd)

Determine the probability that you flip a coin 10 times and that you get all heads.

Solution:

- S is too big to enumerate.
- Use the multiplication rule:

$$|S| = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10} = 1{,}024$$
$$|E| = 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1^{10} = 1$$
$$\text{Prob}(E) = \frac{|E|}{|S|} = \frac{1}{1{,}024} = 0.00098.$$

- Of course, we really know $|E| = 1$ directly, since

$$E = \{(H, H, H, H, H, H, H, H, H, H)\}$$

Terminology and background (cont'd)

Determine the probability that you flip a coin three times and that you get exactly two heads.

Solution:

$$S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$$
$$E = \{(H,H,T), (H,T,H), (T,H,H)\}$$
$$\text{Prob}(E) = \frac{|E|}{|S|} = \frac{3}{8} = 0.375\,.$$

Terminology and background (cont'd)

Determine the probability that you flip a coin ten times and that you get exactly two heads.

This is an example of a Bernoulli trial. We'll look at this later.

Terminology and background (cont'd)

Given a standard deck of cards, which is the probability of drawing one card and that card being a 2 or a 3?

Solution:

$$|S| = 52,$$
$$E = \{2\clubsuit, 2\diamondsuit, 2\heartsuit, 2\spadesuit, 3\clubsuit, 3\diamondsuit, 3\heartsuit, 3\spadesuit\}, \text{ so that } |E| = 8,$$
$$\text{Prob}(E) = \tfrac{8}{52} = \tfrac{2}{13} = 0.154.$$

Complement

- If E is an event in S, then its *complement* is given by

$$E' = S - E\,.$$

- Note that

$$E' \cup E = S \qquad \text{and} \qquad E' \cap E = \emptyset.$$

- Probability of a complement?

$$\text{Prob}(E') = 1 - \text{Prob}(E).$$

- Why? Since

$$|E'| = |S| - |E|,$$

  it follows that

$$\text{Prob}(E') = \frac{|E'|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - \text{Prob}(E).$$

Complement (cont'd)

- Probability of a complement is

$$\text{Prob}(E') = 1 - \text{Prob}(E).$$

- Why is this important?
- Sometimes it's easier to find the probability of some event E by
  - computing the probability of its complementary event E', and then
  - computing $\text{Prob}(E) = 1 - \text{Prob}(E')$.

Complement (cont'd)

- **The birthday "paradox":** Given a class with 6 students, what is the probability that at least two students will share the same birth month?
- **Solution:** Let

$$S = \text{"all possible assignments of months to the 6 students"},$$
$$E = \text{"all such assignments, at least one month is repeated"}.$$

- Easy to see that $|S| = 12^6 = 2{,}985{,}984$.
- Calculating $|E|$: seems hard, so calculate $|E'|$ instead:

$$|E'| = P(12, 6) = 12 \times 11 \times 10 \times 9 \times 8 \times 7 = 665{,}280.$$

  Thus

$$\text{Prob}(E') = \frac{|E'|}{|S|} = \frac{665{,}280}{2{,}985{,}984} = 0.2228 = 22.28\%$$

  So

$$\text{Prob}(E) = 1 - \text{Prob}(E') = 1 - 0.2228 = 0.7772 = 77.72\%$$

Elementary rules for probability

- Recall addition and multiplication rules for counting.
- Similarly, there are addition and multiplication rules for probability.
- To properly state these rules, we need two new concepts: *disjointness* and *independence*:
  - Two or more events are *disjoint* if the outcomes associated with one event are not present in the outcomes of any of the other events (i.e., if the events form non-overlapping sets). More formally, two events E_1 and E_2 are *disjoint* if $E_1 \cap E_2 = \emptyset$.
  - Two events E_1 and E_2 are *independent* if the outcome of any one of these events does not *in any way* impact or influence the outcome of the other event. More formally, two events E_1 and E_2 are *independent* if

$$\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2).$$

Elementary rules for probability (cont'd)

- **Example:** A six-sided die is rolled. Are the events "roll an odd number" and "roll an even number" disjoint?
- **Solution:** Let

$$E_1 = \text{"roll an odd number"} = \{1, 3, 5\}$$

and

$$E_2 = \text{"roll an even number"} = \{2, 4, 6\}.$$

Since $E_1 \cap E_2 = \emptyset$ contains no elements, the events are disjoint.

Elementary rules for probability (cont'd)

- **Example:** A six-sided die is rolled. Are the events "roll an odd number" and "roll an even number" independent?
- **Solution:** The events are not independent.
  - Using the informal criterion: If we know that event E_1 has occurred, then event E_2 can *never* occur. Moreover, if E_1 has not occurred, then E_2 *must* occur. Since the outcome of E_1 influences the outcome of E_2, the events E_1 and E_2 are not independent.
  - Using the formal criterion:
    - We know that $\text{Prob}(E_1) = \frac{3}{6} = \frac{1}{2}$ and $\text{Prob}(E_2) = \frac{3}{6} = \frac{1}{2}$. Hence $\text{Prob}(E_1) \cdot \text{Prob}(E_2) = \frac{1}{4}$.
    - Since $E_1 \cap E_2 = \emptyset$, we find that $\text{Prob}(E_1 \cap E_2) = 0$.
    - So $\text{Prob}(E_1) \cdot \text{Prob}(E_2) \neq \text{Prob}(E_1 \cap E_2)$.

Second part establishes a useful general result: *disjoint events (having non-zero probabilities) are never independent*.

Elementary rules for probability (cont'd)

- **Example:** Two cards are drawn from a fresh deck without replacement. Are the outcomes associated with each draw disjoint?
- **Solution:** Let E_1 and E_2 denote the events of drawing the first and second cards from the deck. Clearly $E_1 \cap E_2$ consists of 51 possibilities. Thus $E_1 \cap E_2 \neq \emptyset$, and so the outcomes are *not* disjoint.

Elementary rules for probability (cont'd)

- **Example:** Two cards are drawn from a fresh deck without replacement. Are the outcomes associated with each draw independent?
- **Solution:** The events are not independent.

  Informal criterion:
  - The outcome of E_1 has some influence on E_2.
  - For example, if A♠ is drawn on the first draw then it cannot be drawn on the second draw.

Elementary rules for probability (cont'd)

- **Example:** Two cards are drawn from a fresh deck without replacement. Are the outcomes associated with each draw independent?
- **Solution:** The events are not independent.

  Formal criterion:
  - Choose a specific outcome, such as E_1 = "drawing A♠" and E_2 = "drawing K♠".
  - We know that $\text{Prob}(E_1 \cap E_2) = \frac{1}{52} \cdot \frac{1}{51} = \frac{1}{2{,}652}$.
  - Clearly, $\text{Prob}(E_1) = \frac{1}{52}$.
  - With no knowledge of E_2, we have $\text{Prob}(E_2) = \frac{1}{52}$.
  - So $\text{Prob}(E_1) \cdot \text{Prob}(E_2) = \frac{1}{52} \cdot \frac{1}{52} = \frac{1}{2{,}704}$.
  - Since $\text{Prob}(E_1 \cap E_2) \neq \text{Prob}(E_1) \cdot \text{Prob}(E_2)$, the events are not independent.

Addition rule for probability

- Let E_1 and E_2 be disjoint events. Then

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2).$$

- **Why?**
  - Since E_1 and E_2 are disjoint,

$$|E_1 \cup E_2| = |E_1| + |E_2|$$

    (recall from chapter on sets).
  - So

$$\begin{aligned}\text{Prob}(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} \\ &= \text{Prob}(E_1) + \text{Prob}(E_2).\end{aligned}$$

Addition rule for probability (cont'd)

- **Example:** What is the probability of drawing a blackjack in the initial deal from a fresh deck?
- **Solution:** Since they are disjoint events, we have

$$\begin{aligned}\text{Prob(blackjack)} = \ &\text{Prob(ace, then value=10)} \\ &+ \text{Prob(value=10, then ace)}.\end{aligned}$$

- We have

$$\begin{aligned}\text{Prob(ace, then value=10)} &= \text{Prob(ace on first draw)} \\ &\quad \times \text{Prob(value=10 on second draw)} \\ &= \tfrac{4}{52} \times \tfrac{16}{51} = 0.024\end{aligned}$$

$$\begin{aligned}\text{Prob(value=10, then ace)} &= \text{Prob(value=10 on first draw)} \\ &\quad \times \text{Prob(ace on second draw)} \\ &= \tfrac{16}{52} \times \tfrac{4}{51} = 0.024\,.\end{aligned}$$

- Thus $\text{Prob(blackjack)} = 0.024 + 0.024 = 0.048$ (or 4.8%).

Addition rule for probability (cont'd)

- **Example:** A person rolls a die and wins a prize if the roll is a 1 or a 2. What is the probability of winning?
- **Solution:** The events $E_1 = \{1\}$ and $E_2 = \{2\}$ are disjoint. So

$$\begin{aligned}\text{Prob}(1 \text{ or } 2) &= \text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) \\ &= \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3} = 0.333\,.\end{aligned}$$

- This was easy enough to do directly, since we're interested in $\text{Prob}(E)$ for $E = \{1, 2\}$. So

$$\text{Prob}(E) = \frac{|E|}{|S|} = \tfrac{2}{6} = \tfrac{1}{3} = 0.333\,.$$

Multiplication rule for probability

- If E_1 and E_2 are independent events, then

$$\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2)$$

- **Why?** This is simply the formal definition of independence.

Multiplication rule for probability (cont'd)

- **Example:** Determine the probability that you flip a coin three times and that you get exactly three heads.
- **Solution:** Let

$$E = \text{“flip coin three times and get all heads”}$$

- Constituent events

$$E_i = \text{“toss \# } i \text{ is a head”} \qquad (i = 1, 2, 3)$$

  are intuitively independent, and so

$$\begin{aligned}\text{Prob}(E) &= \text{Prob}(E_1 \cap E_2 \cap E_3) \\ &= \text{Prob}(E_1) \cdot \text{Prob}(E_2) \cdot \text{Prob}(E_3)\,.\end{aligned}$$

- Clearly

$$\text{Prob}(E_1) = \text{Prob}(E_2) = \text{Prob}(E_3) = \tfrac{1}{2}.$$

- So

$$\text{Prob}(E) = \tfrac{1}{8} = 0.125.$$

Multiplication rule for probability (cont'd)

- **Example:** Buying a car, with the following options:
  - color (8 choices, including red)
  - A/C (yes or no)
  - 4WD (yes or no)

  Can't decide, so choose randomly. What is the probability that you will wind up with a red car, with air-conditioning, but without the 4-wheel drive option?

Multiplication rule for probability (cont'd)

- **Solution:** Let E denote the event of interest (red car, A/C, no 4WD). Constituent events are

$$E_1 = \text{"choose red color"},$$
$$E_2 = \text{"choose A/C option"},$$
$$E_3 = \text{"don't choose 4WD option"}.$$

with corresponding sample spaces of sizes

$$|S_1| = 8, |S_2| = 2, |S_3| = 2.$$

- Since $|E_1| = |E_2| = |E_3| = 1$, we have

$$\text{Prob}(E_1) = \tfrac{1}{8}, \text{Prob}(E_2) = \tfrac{1}{2}, \text{Prob}(E_3) = \tfrac{1}{2}.$$

- Since the events are intuitively independent, we have

$$\begin{aligned}\text{Prob}(E) &= \text{Prob}(E_1 \cap E_2 \cap E_3) \\ &= \text{Prob}(E_1) \cdot \text{Prob}(E_2) \cdot \text{Prob}(E_3) \\ &= \tfrac{1}{8} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{32} = 0.031.\end{aligned}$$

General rules for probability

- Elementary rules hold only in special cases:
  - Addition rule: events must be disjoint
  - Multiplication rule: events must be independent
- What to do if these don't hold?
- **Example:** Given a standard deck of cards, what is the probability of drawing a red card *or* a 2?
- **Solution:**
  - Let S be the sample space. Then $|S| = 52$.
  - Let E_1 be the event "pick a red card". Then $|E_1| = 26$.
  - Let E_2 be the event "pick a two". Then $|E_2| = 4$.
  - Note that $|E_1 \cup E_2| \neq |E_1| + |E_2| = 30$. Since there are two black 2's and two red 2's, we don't want to double-count the red 2's! So $|E_1 \cup E_2| = 28$.
  - So

$$\text{Prob}(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \tfrac{28}{52} = 0.538.$$

General addition rule for probability

- For *any* two events E_1 and E_2,

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2)$$

- **Why?** Simple consequence of the inclusion/exclusion rule

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

- Note that this reduces to the earlier rule

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) \qquad \text{for disjoint } E_1, E_2$$

General addition rule for probability (cont'd)

- **Example:** Given a standard deck of cards, what is the probability of drawing a red card *or* a 2?
- **Solution:**
  - $|S| = 52$, as before.
  - Let E_1 = "pick a red card". Then $|E_1| = 26$, and so $\text{Prob}(E_1) = \tfrac{26}{52} = \tfrac{1}{2}$.
  - Let E_2 = "pick a 2". Then $|E_2| = 4$, and so $\text{Prob}(E_2) = \tfrac{4}{52} = \tfrac{1}{13}$.
  - Since there are two red 2's, $|E_1 \cap E_2| = 2$, and so $\text{Prob}(E_1 \cap E_2) = \tfrac{2}{52} = \tfrac{1}{26}$.
  - Thus

$$\begin{aligned}\text{Prob}(E_1 \cup E_2) &= \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2) \\ &= \tfrac{1}{2} + \tfrac{1}{13} - \tfrac{1}{26} = \tfrac{7}{13} \\ &= 0.538\end{aligned}$$

General addition rule for probability (cont'd)

- **Example:** I flip a coin and roll a six-sided die. What is the probability that I get a head or roll either a 1 or a 2 ?
- **Solution:**
  - We have

$$S_1 = \{\text{head}, \text{tails}\}, \text{ and so } |S_1| = 2,$$
$$S_2 = \{1, 2, 3, 4, 5, 6\}, \text{ and so } |S_2| = 6,$$
$$E_1 = \text{“flip coin and get head”}, \text{ and so } |E_1| = 1,$$
$$E_2 = \text{“roll die and get 1 or 2”}, \text{ and so } |E_2| = 2.$$

  - Thus

$$\text{Prob}(E_1) = \frac{|E_1|}{|S_1|} = \tfrac{1}{2}$$
$$\text{Prob}(E_2) = \frac{|E_2|}{|S_2|} = \tfrac{2}{6} = \tfrac{1}{3}\,.$$

General addition rule for probability (cont'd)

- **Example:** I flip a coin and roll a six-sided die. What is the probability that I get a head or roll either a 1 or a 2 ?
- **Solution (cont'd):**
  - So far:

$$\text{Prob}(E_1) = \tfrac{1}{2}$$
$$\text{Prob}(E_2) = \tfrac{1}{3}\,.$$

  - To compute $\text{Prob}(E_1 \cap E_2)$, note that E_1 and E_2 are independent. So

$$\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2) = \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{6}.$$

  - Hence

$$\text{Prob}(E_1 \cup E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \cap E_2)$$
$$= \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{6} = \tfrac{2}{3} = 0.667\,.$$

General multiplication rule for probability

- Recall that

$$\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2)$$

  when E_1, E_2 are independent.
- Want to compute $\text{Prob}(E_1 \cap E_2)$ when E_1, E_2 not necessarily independent.
- Need one additional concept: The *conditional probability* of E_1 occurring, *given* that E_2 occurs, is denoted $\text{Prob}(E_1|E_2)$.
- If E_1, E_2 are independent, then $\text{Prob}(E_1|E_2) = \text{Prob}(E_1)$.

General multiplication rule for probability (cont'd)

- **Example:** A standard six-sided die is rolled by your friend, but in such a way that you cannot see what value comes up. Your friend tells you that the value that comes up is odd, but does not tell you what specific value was rolled. What is the probability that she rolled a 3?
- **Solution:**
  - Let E_1 = “a 3 is rolled” and E_2 = “the roll is odd”. Then

$$\text{Prob}(E_1) = \tfrac{1}{6}.$$

  - Since the roll was odd, $E_2 = \{1, 3, 5\}$.
  - Since $|E_2| = 3$ and one of the three elements of E_2 is 3,

$$\text{Prob}(E_1|E_2) = \tfrac{1}{3}.$$

General multiplication rule for probability (cont'd)

- Rule for computing conditional probability:

$$\text{Prob}(E_1|E_2) = \frac{\text{Prob}(E_1 \cap E_2)}{\text{Prob}(E_2)}.$$

- **Why?**
  - Since the event E_2 has happened, think of E_2 as a new sample space.
  - $\text{Prob}(E_1|E_2)$ may be interpreted as the fraction of E_2 that is covered by E_1 elements.
  - But these E_1 elements must really be elements of $E_1 \cap E_2$.

General multiplication rule for probability (cont'd)

- The general multiplication rule is now given by

$$\begin{aligned}\text{Prob}(E_1 \cap E_2) &= \text{Prob}(E_2) \cdot \text{Prob}(E_1|E_2)\\ &= \text{Prob}(E_1) \cdot \text{Prob}(E_2|E_1)\end{aligned}$$

- **Why?**
  - First line follows from definition

$$\text{Prob}(E_1|E_2) = \frac{\text{Prob}(E_1 \cap E_2)}{\text{Prob}(E_2)}$$

    of conditional probability.
  - Second line:

$$\begin{aligned}\text{Prob}(E_1 \cap E_2) &= \text{Prob}(E_2 \cap E_1)\\ &= \text{Prob}(E_1) \cdot \text{Prob}(E_2|E_1)\end{aligned}$$

General multiplication rule for probability (cont'd)

- **Example:** What is the probability of picking 2 aces from a deck of cards?
- **Solution:**
  - Let E_1 = "pick Ace on first draw" and E_2 = "pick Ace on second draw".
  - $\text{Prob}(E_1) = \frac{4}{52}$.
  - E_1 and E_2 are not independent: drawing the first card changes the deck.
  - $\text{Prob}(E_2|E_1) = \frac{3}{51}$.
  - So

$$\begin{aligned}\text{Prob}(E_1 \cap E_2) &= \text{Prob}(E_1) \cdot \text{Prob}(E_2|E_1)\\ &= \tfrac{4}{52} \times \tfrac{3}{51} = 0.0045.\end{aligned}$$

Bernoulli trials and probability distributions

- What is the probability of flipping a coin ten times and getting three heads?
- What is the probability of rolling a pair of dice ten times and getting three 7's?
- Heart of such problems? We're repeating an experiment having two outcomes, say O_1 and O_2.
  - Let $p = \text{Prob}(O_1)$.
  - Then $\text{Prob}(O_2) = 1 - p$.
- Out of n repetitions, we want to know the probability that the first outcome happens k times.

Bernoulli trials and probability distributions (cont'd)

- We have n slots to fill, with k instances of O_1 and $n-k$ of O_2.
- We label each O_1-slot by p, and each O_2-slot by $1-p$.
- The probability of any given choice is $p^k(1-p)^{n-k}$.
- But we have $C(n,k)$ different ways to fill the slots with k instances of O_1 and $n-k$ of O_2.
- So

$$\text{Prob}(O_1 \text{ happens } k \text{ times }) = C(n,k)p^k(1-p)^{n-k}.$$

- This equation is called the *binomial distribution*.
  - Symmetric about its average value (its *mean*).
  - Mean value is np.
  - Variance (a measure of spread) is $np(1-p)$.

Bernoulli trials and probability distributions (cont'd)

- **Example:** What is the probability of flipping a coin 10 times and getting heads exactly twice?
- **Solution:**
  - This is a Bernoulli trial with

$$p = \tfrac{1}{2}, n = 10, k = 2\,.$$

  - So the probability is given by

$$C(10,2)\cdot(\tfrac{1}{2})^2\cdot(\tfrac{1}{2})^8 = 45\cdot(\tfrac{1}{2})^{10} = \tfrac{45}{1024} = 0.0439\,.$$

Bernoulli trials and probability distributions (cont'd)

- **Example:** What is the probability of rolling a pair of dice ten times and getting three 7's?
- **Solution:**
  - This is a Bernoulli trial with

$$p = \tfrac{1}{6}, n = 10, k = 3\,.$$

  - So the probability is given by

$$C(10,3)\cdot(\tfrac{1}{6})^3\cdot(\tfrac{5}{6})^7 = 120\cdot(\tfrac{1}{6})^3\cdot(\tfrac{5}{6})^7 = \tfrac{390{,}625}{2{,}519{,}424} = 0.1550\,.$$

Expected Value

- How much can you expect to win (or lose) on a New York State Pick 6 lottery ticket?
- Need the notion of *expected value* of an event.
- Let E be an event with outcomes $O_1, O_2, \ldots O_n$. Then

$$\text{Expected value of } E = \sum_{j=1}^{n} O_j \cdot \text{Prob}(O_j) =$$
$$O_1\cdot\text{Prob}(O_1) + O_2\cdot\text{Prob}(O_2) + \cdots + O_n\cdot\text{Prob}(O_n).$$

- Takes into account the fact that different outcomes may have different probabilities.

Expected Value (cont'd)

- **Example:** What's the expected value when one tosses a fair six-sided die once?
- **Solution:** The outcomes are

$$O_1 = 1, O_2 = 2, \ldots, O_6 = 6.$$

- Since the die is fair, each outcome is equally-likely. Thus

$$\text{Prob}(O_1) = \text{Prob}(O_2) = \ldots \text{Prob}(O_6) = \tfrac{1}{6}.$$

- So

$$\begin{aligned}\text{Expected value} &= \tfrac{1}{6} \times 1 + \tfrac{1}{6} \times 2 + \tfrac{1}{6} \times 3 + \tfrac{1}{6} \times 4 + \tfrac{1}{6} \times 5 + \tfrac{1}{6} \times 6 \\ &= \tfrac{1}{6} + \tfrac{2}{6} + \tfrac{3}{6} + \tfrac{4}{6} + \tfrac{5}{6} + \tfrac{6}{6} \\ &= \tfrac{21}{6} = 3.5\end{aligned}$$

Expected Value (cont'd)

- **Example:** What's the expected value when one tosses a fair six-sided die once?
- **Solution (cont'd):** Note that in this case, the expected value is also the average of the outcomes $1, 2, 3, 4, 5, 6$.
- That's because the outcomes were equally likely.
- This is true in general: if an event E has outcomes $O_1, \ldots, O_n$ that are equally likely, then the

$$\text{Expected value of } E = \frac{1}{n} \sum_{j=1}^{n} O_j.$$

- Why? Since the n events are equally likely, they each have probability $1/n$.

Expected Value (cont'd)

- **Example:** How much can you expect to win (or lose) playing New York State Pick Six?
- Rules?
  - Pay \$1 to play two games.
  - For each game, you choose 6 numbers out of 59.
  - Winning numbers are chosen by drawing 6 ping-pong balls from a rotating drum.
  - Simplify analysis by only considering the jackpot prize.
  - Let's see how this depends on the jackpot amount a.
  - Further simplification: only one winning ticket.

Expected Value (cont'd)

- **Example:** How much can you expect to win (or lose) playing New York State Pick Six?
- **Solution:**
  - The total number of ways 6 balls can be chosen out of 59 is

$$\begin{aligned}C(59, 6) = \frac{59!}{6!53!} &= \frac{59 \times 58 \times 57 \times 56 \times 55 \times 54}{6 \times 5 \times 4 \times 3 \times 2} \\ &= 59 \times 58 \times 57 \times 7 \times 11 \times 3 \\ &= 45{,}057{,}474\end{aligned}$$

  - Since two games per ticket, the probability of one ticket winning the jackpot is $\frac{2}{45{,}057{,}474} = \frac{1}{22{,}528{,}737} = 4.4 \times 10^{-9}$.
  - The expected amount you would win *if the ticket were free* would be

$$\text{Prob(win)} \times a + \text{Prob(lose)} \times 0 = \frac{a}{22{,}528{,}737}$$

  - Since the ticket costs \$1, the expected winnings/loss for one ticket is

$$\frac{a}{22{,}528{,}737} - 1$$

Expected Value (cont'd)

- **Example:** How much can you expect to win (or lose) playing New York State Pick Six?
- **Solution (cont'd):**
  - If the jackpot amount is a, then the expected winnings/loss for one ticket is
$$\frac{a}{22{,}528{,}737} - 1\,.$$
  - What are the expected payoffs for one ticket for certain amounts of the grand jackpot?

Jackpot amount	Expected payoff
\$1,000,000	−\$0.955612
\$10,000,000	−\$0.556221
\$20,000,000	−\$0.112245
\$100,000,000	\$3.43878

  - Break-even point: \$22,528,737