Beyond *Varsity Math*:
The red-and-blue-balls puzzle
An odds inversion problem
1 Introduction

1.1 Acknowledgements and references

1.2 To the reader

2 The Varsity Math problem

3 Generalizing the problem

3.1 Questions to be answered

3.2 Main results

4 Preliminaries

4.1 Notation

4.2 Setting up the problem

4.2.1 The Diophantine equation

4.2.2 Formal and admissible solutions

4.2.3 Trivial solutions

4.2.4 Symmetry

4.2.5 Change of variables

4.2.6 Existence and completeness of solutions

4.2.7 Mathematica

5 Exploring the problem

5.1 The character of the equation

5.1.1 Plotting the equations
What values of $D$ are possible?

5.2 Trivial solutions

5.2.1 Additional solutions that always exist if $q = 2p - 1$

5.3 Reverse search

5.3.1 Selected results of reverse search

5.4 The “recycling” recurrence

5.4.1 Examples

5.4.2 Elliptical case

5.4.3 When removing one ball does not change the odds

5.4.4 Function to generate recycling series

6 Special cases

6.1 Special case: $p/q = 0$

6.2 Special case: $p/q = 1$

6.3 Special case: $p = 1$ or $2$

7 Parabolic case: $p/q = 1/2$

8 Elliptical case: $p/q > 1/2$

8.1 Method of bracketed direct search

8.1.1 Plotting the elliptical case vs. $z$

8.2 Function to solve elliptic case by bracketed search
Examples

8.2.2 Example of a very elongated ellipse

8.3 Exhaustive enumeration of solutions

8.3.1 Refining the bound
8.3.2 Enumeration of elliptical solutions with $x, y < 999$

8.4 Placing bounds on $p$ and $q$

8.5 Vertex solutions

8.5.1 Symmetry when vertex solutions exist
8.5.2 Midsection solutions
8.5.3 Near-triangular solutions

9 Hyperbolic case: $p/q < 1/2$

9.1 Sign of $u, v$ and admissibility of solutions

9.2 Growth rate of solutions for small $p/q$

10 Hyperbolic case, $D > 0$ square

10.1 Ratios $p/q$ giving $D$ square

10.2 Method of factorization

10.3 Existence and completeness of solutions

10.4 Bound on magnitude of solutions

10.5 Function to solve hyperbolic square $D$ case

10.5.1 Examples: ratios with smallest $p, q$ giving square $D$
Example with larger $q$, $p/q$ close to 1/2

11 Hyperbolic case, $D > 0$ nonsquare

11.1 Continued fractions

11.2 Convergents

11.3 Conversion to Pell equation

11.4 Solution of Pell equation
  
  11.4.1 Example: $p/q = 7/18$

11.5 Functions to solve the Pell equation

11.6 A cautionary example: $p/q = 6/17$

11.7 Function to solve hyperbolic case via Pell Equation

11.8 Solutions from the trivial solutions via Pell recurrence

11.9 Classes of solutions
  
  11.9.1 Solution classes and fundamental solutions
  
  11.9.2 Completeness
  
  11.9.3 Functions for testing if two solutions are in the same class

11.10 Solving hyperbolic case by direct search
  
  11.10.1 Bounds on size of fundamental solution
  
  11.10.2 Function to find fundamental solutions $(u, v)$ by direct search
  
  11.10.3 Function to solve hyperbolic case for $(x, y)$ by search

11.11 Feasibility of search

11.12 Method of solution by reduction of RHS to 1
Trivial solutions are found by this method

11.12.2 Example: $p/q = 4/11$ solved manually by method of reduction

11.12.3 Function to solve for $(u, v)$ by method of reduction

11.12.4 Examples

11.12.5 Function to reduce set of solutions to representatives of distinct classes

11.12.6 Examples

11.12.7 Function to solve hyperbolic case for $(x, y)$ by reduction method

11.12.8 Examples

11.13 Method of solution by recursive reduction of RHS

11.13.1 Example: $p/q = 5/11$ solved manually by method of recursive reduction

11.13.2 Putting the method of recursive reduction into a function

11.14 Prime values of $p$ are special

11.14.1 The case $p = 4$ is also special

11.15 Near-triangular solutions for hyperbolic cases

12 Various proofs

12.1 No solutions exist between the three ellipse circum-vertex points

12.2 Number of solutions for hyperbolic nonsquare $D$ is infinite

12.3 Pell recurrence from trivial solutions gives recycling triplets

12.3.1 Base case

12.3.2 Inductive step

12.4 Recycling recurrence is complete for $p = 1$ and $p = 2$
12.5 If $p > 2$ is prime, the only solution classes are the trivial-solution classes

12.5.1 Introduction
12.5.2 Rewriting Equation (11) in factored form with $p$ on RHS only
12.5.3 The trivial classes give a term that is a multiple of $p^2$
12.5.4 If one term is a multiple of $p^2$, the solution must be in a trivial class
12.5.5 Both terms cannot be divisible by $p$
12.5.6 Generalizing the partitioning constraint

12.6 If $p = 4$, the only solution classes are the trivial-solution classes

12.6.1 Proof that the relatively prime solutions for $p = 4$ belong to the trivial-solution classes
12.6.2 Proof that for $p = 4$ there are no solutions with $\gcd(u, v) = 2$

13 Solving using Mathematica

14 Open questions

14.1 Special cases
14.2 Properties of the recycling recurrence